A Class of Tractable Incomplete-Market Models for Studying Asset Returns and Risk Exposure*

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Abstract

We present a class of tractable incomplete-market models, where agents face both aggregate risk and limited participation in financial markets. Tractability relies on the assumptions of small asset volumes and of a period utility function that is linear beyond a threshold, in line with Fishburn's (1977) contribution in decision theory. We prove the existence of an equilibrium and derive theoretical results regarding asset prices and consumption choices. This small-trade model is able to reproduce a low safe return and a high equity premium, together with a realistic representation of household exposure to both idiosyncratic and aggregate risks.

Keywords: Incomplete markets, risk sharing, consumption inequalities.

JEL codes: E21, E44, D91, D31.

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1 Introduction

Infinite-horizon incomplete insurance-market models with credit constraints are known to be difficult to solve in the presence of aggregate shocks. These models generate a large amount of heterogeneity, reflected by the time-varying distribution of agents' wealth with large support, meaning that numerical methods are required to approximate the equilibrium (Krusell and Smith 1998). In this paper, we use a class of incomplete-market models to conduct theoretical investigations of equilibrium allocations and asset prices. We prove the existence of an equilibrium in the presence of aggregate shocks, heterogeneous levels of idiosyncratic risk, and stock-market participation costs, for which we can analytically examine the main determinants of risk allocation and of asset prices.[1] Our modeling strategy is based on two assumptions. First, we assume that the period utility function is linear above a certain threshold, strictly concave below the threshold, but globally smooth and concave. This utility function was first introduced in decision theory by Fishburn (1977) to analyze risk for "below-target returns".[2] The way the utility function provides tractability in incomplete-market models is interesting compared to alternative approaches. In particular, incomplete-market models have often relied on quasi-linearity in the labor supply to reduce the state space dimension (Scheinkman and Weiss 1986; Lagos and Wright 2005; Challe, LeGrand, and Ragot 2013; Le Grand and Ragot 2016; Wen 2015, among others). Assuming linearity in the labor supply has the drawback that the consumption of agents with infinite labor elasticity is constant and independent of wealth and income. This infinite elasticity is also far too high at the household level (see Hall 2010 for a recent survey). The Fishburn utility function, linear beyond a given threshold, may thus be an attractive alternative for studying consumption dynamics and realistic labor income processes, as we do in this paper. Our second assumption is that the supply of securities is not too large. This implies that credit constraints are binding for agents who have experienced only a small number of consecutive bad idiosyncratic

[1] Note that the existence of simple recursive equilibria in such environments is still an open question (Miao, 2006).

[2] This captures the idea that investors are averse to risk for low returns (below a given target), but much less concerned about risk for high returns.

shocks, which generates an equilibrium with a small number of heterogeneous agents. Our economy therefore features a *small-trade* equilibrium, where prices can be analytically studied, as in no-trade equilibria (see below for references), but where we can also investigate consumption allocations and the role of security volumes.

The goal of the paper is to show the usefulness of this setup by theoretically investigating the properties of an environment where two groups of agents face two different labor income processes together with limited participation in financial markets. The motivation for such an environment is based on previous results in the literature. First, it is known that incomplete insurance-markets models can help to solve some asset pricing puzzles but that they are generally unable to reproduce a high equity premium for a realistic calibration (Krusell and Smith 1998; Krusell, Mukoyama, and Smith 2011, among many others). Second, adding limited participation in financial markets can help to reproduce relevant aspects of asset prices (Allen and Gale 1994; Guvenen 2009) and is consistent with empirical evidence (Bricker et al. 2014). Third, empirical investigations of income risks in the US show that high-income households face lower risk than low-income households, who generally do not participate in financial markets.[3]

We derive two main sets of results. First, we prove the existence of an equilibrium and exhibit the structure of this limited-heterogeneity equilibrium. We then theoretically show how the model can generate a low return for the safe asset and a high equity premium. We also characterize the effects of risks and volumes on asset prices, deriving explicit formula to identify all effects at stake. In particular, a higher volume of securities reduces asset prices and improves consumption smoothing, whereas a higher level of idiosyncratic risk generates both a decrease in the bond interest rate and an increase in stock prices.

Second, the calibrated model satisfactorily reproduces household risk exposures and asset price properties. The model generates a more volatile consumption growth rate for low-income households than for high-income households, as in the data. High-income households are also found to bear a larger fraction of the aggregate risk than low income

[3]See De Giorgi and Gambetti (2017), Gârleanu, Kogan, and Panageas (2012), Meyer and Sullivan (2013), and Bricker et al. (2014), among others, for empirical evidence.

households (Parker and Vissing-Jorgensen 2009), while the latter face a larger total risk.

The paper contributes to the theoretical literature on incomplete-market models. In this literature, analytical tractability can be obtained in a no-trade equilibrium, as in Constantinides and Duffie (1996) or Krusell, Mukoyama, and Smith (2011). In these economies, assets can be priced, even in absence of trade. In our model, trades do actually occur at the equilibrium. We show that limited asset market participation is sufficient to explain both a high equity premium and the volatility of consumption, with a risk aversion as low as 1 in the concave part of our benchmark calibration. Our assumption of a concave-linear utility function is similar to several papers that consider linearity in consumption utility, in leisure utility, or in the production function in order to reduce ex-post heterogeneity, such as Scheinkman and Weiss (1986), Lagos and Wright (2005), Kiyotaki and Moore (2005, 2008), Dang, Holmstrom, Gorton, and Ordoñez (2017), Wen (2015), or Miao and Wang (2017), among others. Finally, this paper generalizes previous work on small-trade models (Challe, LeGrand, and Ragot 2013, Challe and Ragot 2014, or LeGrand and Ragot 2016). The paper is also related to the vast literature on asset prices with heterogeneous agents. Our contribution to this literature is to analyze the interaction of two frictions in a tractable framework: limited participation and incomplete insurance markets with heterogeneous income-risk exposure.[4]

The remainder of the paper is organized as follows. In Section 2, we present the model and derive our equilibrium existence result. In Section 3, we present the intuition underlying our model in simplified versions of our framework. In Section 4, we perform a quantitative exercise to show that the model can reproduce household risk exposures and asset returns. Section 5 discusses the key assumptions of the model. Section 6 concludes.

[4]Among the recent quantitative papers, Guvenen (2009) studies a model with limited participation and household heterogeneity in intertemporal elasticities of substitution. Constantinides and Ghosh (2017) build on Constantinides and Duffie (1996) to construct a no-trade equilibrium with Epstein-Zin preferences. Chien, Cole, and Lustig (2011, 2012) consider an incomplete-market model featuring exogenous trading restrictions, which can easily be simulated. Gomes and Michaelides (2008) and Favilukis (2013) consider models with preference heterogeneity (in terms of intertemporal elasticity of substitution or bequest motive) together with incomplete markets or limited participation.

2 The model

The model relies on three assumptions: (i) incomplete insurance markets, (ii) limited stock market participation, and (iii) a concave-linear utility function.

2.1 Risks and securities

Time is discrete and indexed by $t = 0, 1, \ldots$ The economy is populated by two types of infinitely-lived and ex-ante different agents. Each population of type $i = 1, 2$ is of size 1 and is distributed on a segment J_i according to a measure ℓ_i.[5] We call these two populations "type-1" and "type-2" agents, respectively, and the letter i will consistently refer to the agent's type (1 or 2). The two populations differ only with respect to the severity of their idiosyncratic risk.[6]

2.1.1 Aggregate risk

There is a single aggregate shock $(z_t)_{t\geq 0}$, which can take n different values in the set $Z = \{z_1, \ldots, z_n\}$. The aggregate risk process $(z_t)_{t\geq 0}$ is a time-homogeneous first-order Markov chain with transition matrix $\Pi = (\pi_{kj})_{k,j=1,\ldots,n}$. The probability π_{kj} of moving from state k to state j is thus constant. For every date $t \geq 0$, $z^t \in Z^{t+1}$ denotes a possible history of aggregate shocks up to date t.

2.1.2 Idiosyncratic risk

Agents face an idiosyncratic risk in addition to the aforementioned aggregate risk. This individual risk can be neither avoided nor insured. We call this a productivity risk, although it may cover many other individual risks (such as the risks of unemployment, income, health, etc.) that are likely to affect the agent's productivity (see Chatterjee,

[5] Among others, Feldman and Gilles (1985) have identified issues when applying the law of large numbers to a continuum of random variables. Green (1994) describes a construction of the sets J_i and of the non-atomic measures ℓ_i to ensure that our statements hold. Feldman and Gilles (1985), Judd (1985), and Uhlig (1996) propose other solutions. From now on, we assume that the law of large numbers applies.

[6] Our results could easily be generalized to the case where we assume different masses for both types of agents.

Corbae, Nakajima, and Rios-Rull 2007 for a quantitative discussion). At any point in time, type-i agents can either be *productive* (denoted herein by p), earning income $\omega^i(z_t)$, or *unproductive* (denoted by u), earning income δ^i.[7] Both incomes may depend on the agent's type i. To simplify the exposition, we assume that δ^i does not depend on the aggregate risk z_t, however all of our results can easily be extended to stochastic incomes δ^i. We assume that, regardless of the aggregate state, $\omega^i(z_t)$ is greater than δ^i for both agent types. Moreover, when productive, type-1 agents have a higher income than type-2 agents. These assumptions are summarized in Assumption C below.

For each type-i agent at any date t, the function $\xi_t^i(z^t)$ characterizes the current status of the agent's productivity, taking the value 1 when the agent is productive and 0 when unproductive. For both agents, the productivity risk process $(\xi_t^i(z^t))_{t\geq 0}$ is a two-state process with transition matrix $T_t^i = \begin{pmatrix} \alpha_t^i(z^t) & 1-\alpha_t^i(z^t) \\ 1-\rho_t^i(z^t) & \rho_t^i(z^t) \end{pmatrix}$.

We call $\eta_t^i \in (0,1)$ the share of productive agents in a type-i population. Initial values η_0^1 and η_0^2 being given, the laws of motion of productive shares are:

$$\eta_t^i(z^t) = \alpha_t^i(z^t)\eta_{t-1}^i(z^{t-1}) + (1-\rho_t^i(z^t))(1-\eta_{t-1}^i(z^{t-1})), \text{ for } i = 1,2 \text{ and } t \geq 1. \tag{1}$$

To obtain a tractable framework, we impose the following constraint:

Assumption A (Population shares) *The probabilities α_t^i and the shares η_t^i depend solely on the current aggregate state and not on the whole history. Formally, we have: $\alpha_t^i(z^t) = \alpha^i(z_{t-1})$ and $\eta_t^i(z^t) = \eta^i(z_t)$ for $t \geq 0$ and $i = 1,2$.*

This assumption simplifies the dynamics of the population structure, but does not guarantee analytical tractability, since in general, it does not prevent the wealth distribution from having an infinite support. Assumption A includes the standard case where α_t^i and η_t^i $(i = 1,2)$ are constant. Furthermore, Assumption A implies that the primitives of our model are α_t^i and η_t^i, while transition rates $(\rho_t^i)_{t\geq 0}$ adjust for equation (1) to hold.[8]

[7] Our idiosyncratic productivity risk is similar to Kiyotaki and Moore (2005, 2008), Kocherlakota (2009), and Miao and Wang (2017), although our model and the scope of our paper are very different.

[8] Note that in that case, ρ_t^i will depend on two consecutive states and $(\xi_t^i(z^t))_{t\geq 0}$ is a second-order

2.1.3 Asset markets

There are two types of assets in the economy – a risky stock and a riskless bond, issued by the government. This is the simplest environment in which to study the price of the safe asset and the market price of risk.

The risky asset. There is a constant mass V_X of a Lucas tree. The tree dividend is stochastic and the payoff in state k is y_k ($k = 1, \ldots, n$). At any date t, we denote by P_t the (endogenous) price of one "stock" or "risky asset" (i.e., a share of the tree).

The bond. There is also a riskless bond of maturity one. Purchased at date t at price Q_t, these bonds pay off one unit of the consumption good at the next date in all states of the world. The total supply of bonds is constant and equal to V_B. These bonds – or safe assets – are issued by the government and funded by taxes on productive agents.

Participation structure. We assume that trading stocks requires type-i agents to pay a per period lump-sum participation cost χ_i, while bond trading is free. This cost is consistent with many empirical studies, such as Mankiw and Zeldes (1991) or Vissing-Jorgensen (2002), and is frequently used to generate limited participation.[9] The participation cost is a shortcut for both monetary and non-monetary hurdles to stock market participation. In particular, non-monetary costs may cover informational aspects, such as acquiring and maintaining financial literacy and maintaining an up-to-date knowledge of stock markets. There is strong evidence that the population is heterogeneous in terms of its financial literacy and that people with low financial literacy participate less in stock markets (see van Rooij, Lusardi, and Alessie 2011 among others). The goal of this participation cost is not to provide a fully fledged theory of limited participation, but to quantify the opportunity cost of not participating in financial markets. We make the following assumption in order to define asset market structure.

Assumption B (Participation costs) *We assume that χ_2 is large enough for type-2 agents not to trade stocks, while type-1 agents do not pay participation costs: $\chi_1 = 0$.*

Markov chain.

[9] The impact of participation costs on asset prices has, for instance, been studied in Basak and Cuoco (1998), Heaton and Lucas (1999), Polkovnichenko (2004), Gomes and Michaelides (2008), Guvenen (2009), Walentin (2010), and Favilukis (2013).

This assumption implies that, consistent with the empirical facts presented in Section 4 below, type-1 agents will trade stocks, while type-2 agents will not. Equation (29) in Appendix A provides an explicit formula for an upper bound on χ_2.

The State budget. Bond issuances are financed by income taxes. We assume that there is no government consumption, such that taxes and new bond issuances exactly cover maturing bond payoffs. Moreover, the tax on both types of productive agents is assumed to be proportional to their income. We denote by τ_t the income tax rate, which is independent of the agent's type. A balanced government budget constraint at any date t therefore implies that the tax rate is:

$$\tau_t = \frac{(1 - Q_t)V_B}{\omega_t^1 \eta_t^1 + \omega_t^2 \eta_t^2}. \tag{2}$$

2.2 Agents' preferences

Agents' preferences are a crucial feature of the model's tractability, enabling us to derive our small-trade equilibrium with a finite number of states.

Description of preferences. The period utility function $\tilde{u}$ is smooth, continuous, strictly increasing, and globally concave. It is strictly concave for low values of consumption and may have two linear parts. This assumption can be formally written in terms of the conditions imposed on marginal utility $\tilde{u}'$:

$$\tilde{u}'(c) = \begin{cases} u'(c) & \text{if } c \leq c_1^*, \\ \lambda^2 & \text{if } c_2^* \leq c \leq c_3^*, \\ \lambda^1 \leq \lambda^2 & \text{if } c_4^* \leq c \leq c_5^*, \end{cases} \tag{3}$$

where $0 < c_1^* < c_2^* < c_3^* < c_4^* < c_5^*$. When agents consume a low amount, they value their consumption with the marginal utility $u'(\cdot)$, which is the derivative of a function $u : \mathbb{R}^+ \to \mathbb{R}$ assumed to be twice derivable, strictly increasing, and strictly concave. When agents consume a higher amount, their marginal utility is constant for two consumption intervals, where it is equal to λ^1 or λ^2. The derivative of $\tilde{u}$ is not further specified. As the intervals specified above are disconnected, one can easily construct a function $\tilde{u}$ that is smooth, continuous, strictly increasing, and globally concave. Figure 1 plots the shape of this period utility function.

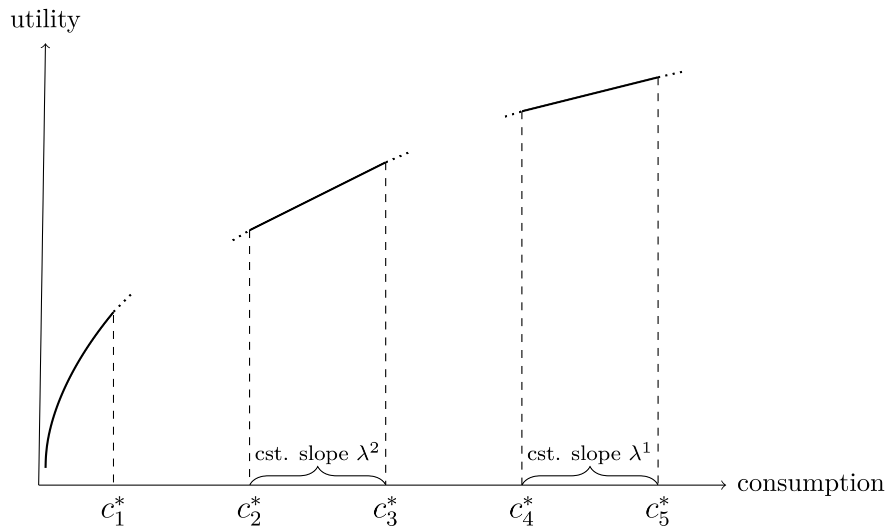

Figure 1: Shape of the period utility function

We now formulate our next assumption.

Assumption C (Income processes) *We assume that in any state* $k = 1, \ldots, n$*, we have* $c_2^* < \omega^2(z_k) < c_3^*$*,* $c_4^* < \omega^1(z_k) < c_5^*$ *and* $\delta^i < c_1^*$ *for* $i = 1, 2$*. This implies that* $\delta^i < \omega^i(z_k)$ *for both types* $i = 1, 2$ *and* $\omega^1(z_k) > \omega^2(z_k)$ $(\forall k = 1, \ldots, n)$*.*

Assumption C states that the income of type 1 and 2 productive agents lies in the set where the utility function is linear, and that the income of type 1 and 2 unproductive agents lies in the set where the utility function is strictly concave. A straightforward corollary is that, in the absence of trade, unproductive type-i agents are endowed with marginal utility $u'(\delta^i)$, while productive type-i agents have marginal utility $\lambda^i < u'(\delta^i)$, where this last inequality illustrates that for any type i, you will be better off if you are productive than if you are unproductive.

Effect on the equilibrium. The assumption of constant marginal utility for productive agents helps generate a limited heterogeneity equilibrium. The individual histories of productive agents are not relevant for the pricing of securities since their marginal utility depends only on their type and not on their past saving choices.

Interpretation. The utility function in equation (3) generalizes the utility function of Fishburn (1977), which is linear above a given threshold and strictly concave below it. In a portfolio choice problem, the agent endowed with such a utility is risk-neutral for large payoffs and risk-averse for low ones. Loosely speaking, this functional form reflects the asymmetry in risk perception. Payoff realizations that are lower than a given threshold are perceived as actual risks, while payoffs greater than the threshold are perceived as being "nice surprises". As explained by Fishburn (1977, p. 123), this concave-linear functional form is "motivated by the observation that decision makers in investment contexts frequently associate risk with failure to attain a target return". In our paper, the concave-linear utility function can be understood as an approximation, according to which the present wealth of productive agents is scarcely affected by saving choices, implying a constant marginal utility. Conversely, unproductive agents use a strictly concave function to value their (lower) consumption. Importantly, this assumption does not imply that productive agents are risk-neutral: the concave part of the function directly affects asset pricing and the agents' behavior (see Section 3.2). We discuss the concave-linear utility further in Section 3.3.

2.3 The agent's program

Timing. At the beginning of every period, the agent observes her current productivity status and dividend payoff.
Allocations. Due to the timing of the agent's program, the agent's choices – consumption levels $(c_t^i)_{t\geq 0}$ and demands for stocks and bonds, respectively denoted by $(x_t^i)_{t\geq 0}$ and $(b_t^i)_{t\geq 0}$ – at date t are mappings defined over the state space of possible shock histories $Z^t \times E^t$.[10] The consumption of any type-i agent is assumed to be positive:

$$\forall t \geq 0,\ c_t^i \geq 0. \tag{4}$$

[10]For the sake of clarity, we drop dependence in the shock histories.

Budget and borrowing constraints. The choices of a type-i agent are limited by a budget constraint in which total resources, made up of income, stock dividends, and security-sale values, are used to consume, pay taxes, and purchase securities. Formally, for all $t \geq 0$:

$$c_t^i + P_t\, x_t^i + Q_t b_t^i + \chi_i 1_{x_t^i > 0} = (1 - \xi_t^i)\delta^i + \xi_t^i \omega_t^i (1 - \tau_t^i) + (P_t + y_t)x_{t-1}^i + b_{t-1}^i, \qquad (5)$$

where $1_{x_t^i > 0}$ is the indicator function equal to 1 if the agent purchases stocks (i.e., if $x_t^i > 0$), and equal to 0 otherwise.

In addition, agents face borrowing constraints. They can neither hold a negative share of stocks nor short-sell the bond. This implies that:[11]

$$\forall t \geq 0,\; x_t^i, b_t^i \geq 0. \qquad (6)$$

A *feasible allocation* for a type-i agent is a collection of plans $(c_t^i, x_t^i, b_t^i)_{t \geq 0}$ such that equations (5) and (6) hold at any date t. The *set of feasible allocations* $\mathcal{A}^i$ is

$$\mathcal{A}^i = \left\{ (c_t^i, x_t^i, b_t^i)_{t \geq 0} :\; \text{equations (4), (5), and (6) hold} \right\}. \qquad (7)$$

Agent's program. The agent's program consists in finding a feasible allocation in the set $\mathcal{A}^i$ that maximizes her intertemporal utility subject to a transversality condition. Instantaneous utilities are discounted by a time preference parameter $\beta \in (0, 1)$. The operator $E_0[\cdot]$ is the unconditional expectation regarding aggregate and idiosyncratic shocks. The

[11] It would be possible to have strictly negative (but not too loose) borrowing constraints on bonds and stocks, while preserving the existence of an equilibrium. However, the set $\mathcal{V}$ of admissible security volumes compatible with the existence of an equilibrium and defined below in Proposition 1 would be different.

initial financial asset endowments are denoted by x^i_{-1} and b^i_{-1}. Formally:

$$\max_{(c^i_t,x^i_t,b^i_t)_{t\geq 0}\in\mathcal{A}^i} E_0\left[\sum_{t=0}^{\infty}\beta^t\tilde{u}(c^i_t)\right] \tag{8}$$

$$\text{s.t. } \lim_{t\to\infty}\beta^t E_0\left[\tilde{u}'(c^i_t)x^i_t\right] = \lim_{t\to\infty}\beta^t E_0\left[\tilde{u}'(c^i_t)b^i_t\right] = 0,$$

$$\{x^i_{-1}, b^i_{-1}, \xi^i_0, z_0\} \text{ are given.}$$

Agents' risk-sharing is limited in three respects. First, as in the Bewley-Huggett-Aiyagari literature, individual risk is uninsurable because no asset is contingent on productivity status. Second, agents face participation and borrowing constraints. Finally, the insurance market against the aggregate shock is potentially incomplete.

2.4 Equilibrium definition

We start with security market clearing conditions, stating that aggregate demand should be equal to total supply, which amounts to V_X for stocks and V_B for bonds. We define the probability measure Λ^i_t, which describes the distribution of type-i agents as a function of their security holdings and the history of their individual status.[12] The market-clearing conditions can therefore be written as:

$$\sum_{i=1,2}\int_{\mathbb{R}^2\times E^t} x\Lambda^i_t(dx,db,d\xi) = V_X, \text{ and } \sum_{i=1,2}\int_{\mathbb{R}^2\times E^t} b\Lambda^i_t(dx,db,d\xi) = V_B. \tag{9}$$

Finally, by Walras' law, the good market clears when asset markets clear.

We can now define a sequential competitive equilibrium.

Definition 1 (Sequential competitive equilibrium) *A sequential competitive equilibrium is a collection of allocations $(c^i_t, x^i_t, b^i_t)_{t\geq 0}$ for $i = 1, 2$ and of price processes $(P_t, Q_t)_{t\geq 0}$ such that, for an initial distribution of stock and bond holdings and of idiosyncratic and aggregate shocks $\{(x^i_{-1}, b^i_{-1}, \xi^i_0)_{i=1,2}, z_0\}$, we have:*

[12] More precisely, $\Lambda^i_t : \mathcal{B}(\mathbb{R})^2 \times \mathcal{B}(E^t) \to [0, 1]$, where for any metric space X, $\mathcal{B}(X)$ denotes the borel sets of X. As an example, $\Lambda^i_t(X, B^S, I)$ (with $(X, B^S, I) \in \mathcal{B}(\mathbb{R})^2 \times \mathcal{B}(E^t)$) is the measure of type-i agents, with holdings in risky assets $x \in X$, in bonds $b \in B^S$, and with an individual history $\xi \in I$.

1. *given prices, individual strategies solve the agents' optimization program in (8);*

2. *the security markets clear at all dates: for any $t \geq 0$, the equations in (9) hold; and*

3. *the probability measures Λ_t^i evolve consistently with individual strategies in each period.*

3 Equilibrium existence and asset price properties

In standard economies featuring uninsurable idiosyncratic shocks, credit constraints, and aggregate shocks, the equilibrium cannot be explicitly derived since it involves a large distribution of agents' wealth. The usual strategy follows Krusell and Smith (1998) by computing approximate equilibria assuming a recursive structure. However, as pointed out by Heathcote, Stroresletten, and Violante (2009), the existence of such an equilibrium is still an open question, despite Miao (2006) and Cao (2016) having proved the existence of *generalized* recursive equilibria in this setup. In our economy, we are able to provide some new results. Proposition 1 states the existence of an equilibrium and characterizes its main theoretical properties.

Proposition 1 (Equilibrium existence) *We assume that:*

$$\forall k \in \{1, \ldots, n\},\ \beta \left(\alpha^1(z_k) + (1 - \alpha^1(z_k)) \frac{u'(\delta^1)}{\lambda^1} \right) < 1. \tag{10}$$

If security volumes (V_B, V_X) belong to a set $\mathcal{V} \subset \mathbb{R}_+ \times \mathbb{R}_+$ containing $(0, 0)$ then an equilibrium with the following features exists:

1. *the end-of-period security holdings of unproductive type-1 and type-2 agents is 0 for both risky and riskless assets;*

2. *the end-of-period stock holdings of type-2 agents is always 0;*

3. *the end-of-period security holdings of productive agents depend only on their type (1 or 2) and the current aggregate state; and*

4. security prices depend solely on the current aggregate state.

To prove equilibrium existence, we rely on the first-order conditions of the agents' program, as in Coleman (1991). We derive these first-order conditions by following the steps of the Theorem 4.15 proof in Stokey and Lucas (1989). Indeed, we cannot use the Kuhn-Tucker theorem, as it assumes the space of allocations to be a Hermitian space and this is not the case for the set of bounded real sequences (to which (c_t^i), (e_t^i), (x_t^i), and (b_t^i) belong). The details of the proof can be found in the Section B Appendix.

The equilibrium exists under two conditions. The first, $\beta(\alpha^1(z_k)+(1-\alpha^1(z_k))\frac{u'(\delta^1)}{\lambda^1}) < 1$, ensures that stock prices are well-defined. If the condition does not hold, the stock price can potentially be infinite because agents are too patient or their self-insurance motive is too strong. This condition has a straightforward formal interpretation. Indeed, first-order conditions yield an Euler equation, expressing the stock price P_t as $P_t = \beta E_t\left[(\alpha_t^1 + (1-\alpha_t^1)u'(\delta_{t+1}^1))(P_{t+1} + y_{t+1})\right]$. The condition $\beta(\alpha^1(z_k)+(1-\alpha^1(z_k))\frac{u'(\delta^1)}{\lambda^1}) < 1$ ensures that the mapping $(P_t)_t \mapsto \left(\beta E_t\left[(\alpha_t^1 + (1-\alpha_t^1)u'(\delta_{t+1}^1))(P_{t+1} + y_{t+1})\right]\right)_t$ is a contraction with a modulus strictly smaller than one. The Banach fixed-point theorem then implies that the asset price is well-defined and finite. Of note, such a condition does not appear in economies with only short-lived assets, which are simpler in this respect. The second condition for equilibrium existence is that security volumes are restricted to belong to a set $\mathcal{V}$, that includes the zero volume case $V_X = V_B = 0$. This assumption implies that security volumes should not be too high, which guarantees that agents are credit-constrained when they become unproductive, as discussed below. The detailed construction of the set $\mathcal{V}$ is provided in the Appendix.

We now discuss further the four points of Proposition 1. A first feature of the equilibrium (point 1) is that productive agents hold securities while unproductive agents of both types do not. Productive agents purchase securities for a standard consumption-smoothing motive but also for a precautionary saving motive since they need to hedge themselves against the risk of becoming unproductive in the next period. Since both securities are in scarce supply – which is reflected in the set $\mathcal{V}$ – security prices are too high for unproductive agents to be able to hold any of these assets. In fact, given these high prices and the expectation that they are likely to be productive, and thus better off, in the future, unproductive agents would like to short securities in order to smooth their consumption. However, credit constraints prevent them from doing so and they simply hold no assets.

A second aspect of Proposition 1 (point 2) is the market segmentation for stocks: only type-1 agents trade stocks, while type-2 do not. This stems from the presence of stock market participation costs. In equilibrium, type-1 agents are rich enough to pay the participation cost and therefore hold stocks. Conversely, type-2 agents are too poor to pay the cost and therefore cannot trade stocks.

A third feature of our equilibrium (point 3) is that the saving choices of productive agents depend solely on the current aggregate state and on the agent's type. This property relies on the linear-concave utility function, which implies that all productive agents of a given type have the same marginal utility – which is independent of their past choices – and therefore express the same demand for securities. The linear-concave utility function also explains why security prices depend solely on the current aggregate state (point 4).

A consequence of Proposition 1 is that the equilibrium features limited heterogeneity. All productive agents of a given type express the same demand for securities, while none of the unproductive agents hold any securities. Furthermore, the beginning-of-period security holdings of a given agent are either null if she was already unproductive before, or equal to those of a productive agent if she was productive. Consequently, for each agent type, the equilibrium only features four categories of agent, characterized by their productivity status (productive or unproductive) in the current and previous periods.

We simplify our notations using Proposition 1. Since security prices depend only on the current aggregate state, we call P_k the price of the risky asset and Q_k the price of the bond in state z_k $(k = 1, \ldots, n)$. Similarly, we call b_k^i the bond holdings of any productive type-i agent in state z_k $(k = 1, \ldots, n)$. Since type-2 agents do not trade stocks $(x^2 = 0)$, productive agents hold all stocks and $x_k^1 = \frac{V_X}{\eta_k^1}$. The equilibrium is therefore characterized by a finite sequence of $4 \times n$ variables $(b_k^1, b_k^2, P_k, Q_k)_{k=1,\ldots,n}$.

Several existence results can be found in the heterogeneous agent literature. In Huggett (1993), agents trade short-lived riskless bonds in the absence of aggregate shocks. Kuhn (2013) extends Huggett's result to permanent idiosyncratic shocks.[13] Miao (2006), Cao (2016), and Cheridito and Sagredo (2016a and 2016b) prove the existence of an equilibrium in an economy featuring asset trades, credit constraints, and idiosyncratic and aggregate risks with a continuum of agents. They consider an economy with general preferences in which agents can trade one type of short-lived asset (claims on capital). Our existence result concerns a setup with limited participation and both a short- and long-lived asset.[14]

3.1 Asset price properties: Intuitions in simpler setups

Our model features heterogeneous uninsurable individual risk, aggregate risk, and positive security volumes. We now examine, in turn, the role of the different model features.

In this section, and only in this section, we further simplify our setup to make the mechanisms as transparent as possible. In particular, we make the following two assumptions: (i) aggregate risk follows an IID process, and (ii) the productivity transition probabilities α^i and ρ^i are constant.

No idiosyncratic risk. As a first benchmark, we study the case where agents face no idiosyncratic risk (i.e., $\alpha^i = 1$ for $i = 1, 2$). Due to limited participation, only type-1 agents trade stocks, whose constant price P^{NIR} (NIR stands for "No Idiosyncratic Risk") verifies $P^{NIR} = \beta E[P^{NIR} + \tilde{y}]$, where $\tilde{y}$ is the next period's uncertain dividend. The

[13] In their seminal paper, Duffie et al. (1994) consider endowment economies in which a finite number of ex-ante heterogeneous agents face aggregate risks and trade long-lived assets with borrowing constraints. They then prove the existence of ergodic equilibria with a recursive characterization, whose state space includes all endogenous variables (such as prices). In a similar vein, Becker and Zilcha (1997) prove the existence of a stationary equilibrium in a production economy with ex-ante heterogeneous agents facing aggregate risk. Krebs (2006) proves the existence of a no-trade equilibrium in a Krusell-Smith economy. Kubler and Schmedders (2002) prove the existence of a recursive equilibrium with a finite number of agents. These papers consider a finite number of households. This assumption helps to prove existence but makes the analysis of the equilibrium properties more difficult as all shocks are "aggregate". It may explain the wide use of a Bewley-type model with a continuum of agents.

[14] In our setup it would additionally be possible to prove that the sequential competitive equilibrium is also a recursive competitive equilibrium in which the state variables are current aggregate and idiosyncratic shocks and beginning-of-period security holdings for both agent types.

gross average stock return R_s^{NIR} is therefore constant and equal to β^{-1}. The riskless bond is traded by both agents and its price is $Q^{NIR} = \beta$. The riskless gross interest rate R_f^{NIR} is identical to the stock return. The equity premium in this environment is null: $R_s^{NIR} - R_f^{NIR} = 0$. Here, limited participation does not in itself imply a non-zero risk premium. This is related to the fact that, in our setup, security holders are endowed with a constant marginal utility.

Zero volumes. We now assume that agents face heterogeneous but constant transition probabilities across idiosyncratic states. Aggregate risk affects dividends, while the income of productive (ω^i) and unproductive (δ^i) agents is constant. Moreover, we have zero volumes of both riskless and risky securities, henceforth denoted by ZV.

In this economy, the equilibrium features complete asset market segmentation where type-1 agents trade stocks and type-2 agents trade bonds if

$$\kappa^2 > \kappa^1, \tag{11}$$

$$\text{where: } \kappa^i = (1-\alpha^i)(\frac{u'(\delta^i)}{\lambda^i} - 1). \tag{12}$$

The type-1 agents' Euler equation for stocks is $P^{ZV} = \beta(1+\kappa^1)(P^{ZV} + E^{\tilde{z}}[y(\tilde{z})])$, or alternatively:

$$P^{ZV} = \frac{\beta(1+\kappa^1)}{1-\beta(1+\kappa^1)} E^{\tilde{z}}[y(\tilde{z})]. \tag{13}$$

Regarding bonds, condition (11) implies that type-2 agents hold bonds, while they are too expensive for type-1 agents:

$$Q^{ZV} = \beta\left(1+\kappa^2\right), \tag{14}$$

$$Q^{ZV} > \beta\left(1+\kappa^1\right). \tag{15}$$

We can take advantage of this simple zero-volume framework to further examine the existence of an equilibrium. To do this, we need to check that type-2 unproductive agents are not willing to purchase bonds and that type-1 unproductive agents refuse to buy both stocks and bonds.[15] The conditions for type-1 agents can be expressed as:

$$P^{ZV}\frac{1}{\lambda^1}u'(\delta^1) > \beta(1-\rho^1+\rho^1\frac{1}{\lambda^1}u'(\delta^1))(P^{ZV}+E^{\tilde{z}}[y(\tilde{z})]), \tag{16}$$

$$Q^{ZV}\frac{1}{\lambda^1}u'(\delta^1) > \beta(1-\rho^1+\rho^1\frac{1}{\lambda^1}u'(\delta^1)), \tag{17}$$

reflecting the fact that both securities are too expensive for type-1 unproductive agents. Using the bond price expression (14), condition (17) on bonds can be simplified to $(\alpha^2 + (1-\alpha^2)\frac{u'(\delta^2)}{\lambda^2})\frac{1}{\lambda^1}u'(\delta^1) > 1-\rho^1+\rho^1\frac{1}{\lambda^1}u'(\delta^1)$, which is always true when (11) holds and when $u'(\delta^1) > \lambda^1$ (Assumption C). After substituting the stock price expression (13), condition (16) becomes $(\alpha^1 + (1-\alpha^1)\frac{u'(\delta^1)}{\lambda^1})\frac{1}{\lambda^1}u'(\delta^1) > 1-\rho^1+\rho^1\frac{1}{\lambda^1}u'(\delta^1)$, which also always holds due to Assumption C. In brief, unproductive agents are never willing to purchase securities. The assumption of a concave-linear utility function helps simplify the Euler equations, since it avoids the dependence of productive agents' marginal utilities in their individual histories.

To conclude our equilibrium existence analysis, we need to verify whether individual consumption levels correspond to the proper concave or linear regions of the utility function. Because of the zero-volume assumption, the consumption level of type-i unproductive agents is δ^i, while it is ω^i for productive agents. Assumption C ensures that the consumption of unproductive agents remains in the strictly concave region and that the consumption of productive agents remains in the linear one.[16]

We can now state our main result in the zero-volume framework.

Proposition 2 (Zero volumes) *In the zero-volume economy described above, the risk premium is an increasing function of the heterogeneity in productivity risk:*

$$R_s^{ZV} - R_f^{ZV} = \frac{1}{\kappa^1} - \frac{1}{\kappa^2}. \tag{18}$$

[15] The condition regarding the non-participation of type-2 agents in the stock market is mainly a matter of stock market participation cost and is not further analyzed here.

[16] Note that in presence of positive volumes, budget constraints are more significant and these conditions are less likely to hold.

The risk premium in equation (18) is strictly positive and reflects the combination of market segmentation and idiosyncratic risk faced by both agent types. The heterogeneity in productivity risk between the two types of agent is positively correlated with the equity premium. On the one hand, when the self-insurance needs of type-2 agents increase, their precautionary saving motive also increases as does their demand for riskless bonds, causing the riskless return to decrease and the equity premium to rise. On the other hand, when the self-insurance needs of type-1 agents fall, their demand for stocks is reduced and they require a higher risky return to hold these stocks. As a result, the heterogeneous demand for self-insurance – in combination with limited stock market participation – is sufficient to generate a strictly positive risk premium, despite both asset payouts being uncorrelated with agents' marginal utilities (as a result of our utility specification).

Positive volumes. We now relax the assumption of zero volumes. For the sake of simplicity, we assume that both bond and stock volumes are small, which allows us to derive closed-form expressions – as first-order expressions – for the equity premium. In addition, and to simplify the expressions, we assume that incomes are not time-varying: δ^i and ω^i are constant for $i = 1, 2$. Only stock dividends are time-varying. PV stands for positive volume.

Proposition 3 (Positive volumes) *If condition (11) holds, the economy will exhibit the following features:*

- *the equity premium, compared to the ZV case, is augmented by two terms: one reflecting the security supply and the other reflecting the equity risk:*

$$R_s^{PV} - R_f^{PV} \approx \underbrace{R_s^{ZV} - R_f^{ZV}}_{=ZV\ risk\ premium} + \beta(1-\alpha^1)\frac{-\frac{u''(\delta^1)}{\lambda^1}}{\alpha^1 + (1-\alpha^1)\frac{u'(\delta^1)}{\lambda^1}} \tag{19}$$

$$\times \left(\underbrace{\frac{E\left[P^{ZV} + y(\tilde{z})\right]}{P^{ZV}}\left(E_t\left[P^{ZV} + y(\tilde{z})\right]\frac{V_X}{\eta^1} + b^1\right)}_{=wealth\ effect} + \underbrace{\frac{V\left[P^{ZV} + y(\tilde{z})\right]}{P^{ZV}}\frac{V_X}{\eta^1}}_{=liquidation\ premium} \right)$$

- *the bond holdings of productive agents are such that*
  - *either* $b^1 = 0$ *and* $\eta^2 b^2 = V_B$ *in case of (endogenous) complete market separation; or*
  - $b_1 \geq 0$ *and* $b_2 \geq 0$. *Bond holdings are then determined by idiosyncratic risk heterogeneity and security volumes:*

$$\eta^2(\mu^1 + \mu^2)b^2 \approx \kappa^2 - \kappa^1 + \mu^1(E^{\tilde{z}}[P^{ZV} + y(\tilde{z})]V_X + V_B), \tag{20}$$

$$\eta^1(\mu^1 + \mu^2)b^1 \approx \kappa^2 - \kappa^1 - \mu^1 E^{\tilde{z}}[P^{ZV} + y(\tilde{z})]V_X + \mu^2 V_B, \tag{21}$$

$$\textit{with: } \mu^i = -(1-\alpha^i)\frac{u''(\delta^i)}{\eta^i \lambda^i} > 0.$$

This proposition illustrates the effect of positive asset volumes on the equity premium and on bond trading. Regarding the equity premium, equation (19) shows that the equity premium is the sum of three terms. The first term $R_s^{ZV} - R_f^{ZV}$ is the equity premium for zero volumes – described above – and reflects market segmentation and heterogeneity in precautionary saving motives. The second term, proportional to $E_t\left[P^{ZV} + y(\tilde{z})\right]$, reflects the wealth effect generated by security holdings. When security holdings rise, productive type-1 agents become better insured against productive risk. These agents therefore face a smaller precautionary saving motive, which causes the risky return and the equity premium to rise. The third term, proportional to $V\left[P^{ZV} + y(\tilde{z})\right]$, reflects the liquidation premium related to the co-movement of unproductive consumption with the asset price. If type-1 agents become unproductive and liquidate their portfolio, they will consume the quantity $\delta^1 + \left(P^{ZV} + y(\tilde{z})\right)\frac{V_X}{\eta^1} + b^1$, which will be high when the asset payoff is high and low otherwise. This positive co-movement of asset price and consumption undermines the hedging properties of stock holdings and explains why agents want to be compensated for this liquidation risk. Since bond payoffs are riskless, there is no bond liquidation premium.

Equations (20) and (21) determine the demand for bonds (in the absence of full market segmentation). From (20), we deduce that the bond demand b^2 of type-2 agents is mainly driven by two factors, heterogeneity in idiosyncratic risk and security volumes. The first factor, the heterogeneity in idiosyncratic risk, $-\kappa^2 - \kappa^1-$, reflects the competition between the two agents to hold the same asset. When the productivity risk for type-2 agents compared to type-1 agents increases, the precautionary motive of type-2 agents compared to type-1 agents also rises, and they purchase more bonds. The second determinant – proportional to μ^1- is the total quantity of securities available for self-insurance purposes. When the security supply increases, the bond price falls and type-2 agents can purchase more bonds. For type-1 agents, the intuition is similar except for the role of stock volumes. Type-1 agents can purchase either stocks or bonds, which are therefore partly substitutes. An increase in stock volumes makes stock cheaper and crowds out bonds in favor of stocks for type-1 agents – this is the term proportional to μ^2.

3.2 Equilibrium structure

We now characterize the equilibrium properties in the general case. Because our equilibrium features limited heterogeneity, we can state the next proposition that provides asset price expressions.

Proposition 4 (Equilibrium properties) *Two distinct subsets exist, denoted by* $I_i \subset \{1, \ldots, n\}$ *(*$i = 1, 2$*), characterizing the states of the world in which only type-i agents trade bonds, such that the* $4 \times n$ *variables* $(b_k^1, b_k^2, P_k, Q_k)_{k=1,\ldots,n}$ *defining the equilibrium* *are given by the following $4 \times n$ equations:*

$$P_k = \beta \sum_{j=1}^{n} \pi_{kj}(\alpha_k^1 + (1-\alpha_k^1)\frac{1}{\lambda^1}u'(\delta^1 + (P_j + y_j)\,\frac{V_X}{\eta_k^1} + b_k^1))(P_j + y_j),\ k \in \{1, \ldots, n\}, \tag{22}$$

$$Q_k = \beta \sum_{j=1}^{n} \pi_{kj}(\alpha_k^1 + (1-\alpha_k^1)\frac{1}{\lambda^1}u'(\delta^1 + (P_j + y_j)\,\frac{V_X}{\eta_k^1} + b_k^1)),\ k \in \{1, \ldots, n\} - I_2, \tag{23}$$

$$Q_k = \beta \sum_{j=1}^{n} \pi_{kj}(\alpha_k^2 + (1-\alpha_k^2)\frac{1}{\lambda^2}u'(\delta^2 + b_k^2)),\ k \in \{1, \ldots, n\} - I_1, \tag{24}$$

$$V_B = \eta_k^1 b_k^1 \text{ and } 0 = b_k^2,\ k \in I_1, \tag{25}$$

$$V_B = \eta_k^2 b_k^2 \text{ and } 0 = b_k^1,\ k \in I_2, \tag{26}$$

$$V_B = \eta_k^1 b_k^1 + \eta_k^2 b_k^2,\ k \in \{1, \ldots, n\} - I_1 - I_2. \tag{27}$$

This equilibrium structure can be thought of in two ways. First, we can consider it as a generalization of the no-trade equilibria studied in Krusell, Mukoyama, and Smith (2011), in which we allow for limited participation and for positive traded volume. Second, this equilibrium structure can be considered as a simplification of the general incomplete-market equilibrium, where credit constraints always bind after one period of unemployment. In any case, this limited-heterogeneity equilibrium allows us to study the effect of volume on asset prices. We now discuss the effects.

Our equilibrium is characterized by equalities (22)–(27). The first three sets of Euler equations provide security prices. Due to limited stock market participation, the risky asset price is only defined by the Euler equation (22) of productive type-1 agents, who hold all the stocks. The equation can be interpreted as follows. The price P_k in state k is equal to the discounted value of asset payoffs discounted by the intertemporal marginal rate of substitution (henceforth, MRS). The probability that the agent remains productive in the next period in state j, with marginal utility λ^1 is $\pi_{kj}\alpha_k^1$. Her MRS in that case is thus simply equal to 1, while the asset payoff is $P_j + y_j$. The probability that the agent becomes unproductive in state j is $\pi_{kj}(1-\alpha_k^1)$. When the agent becomes unproductive, she will sell the whole portfolio (and remain credit-constrained) and will be endowed with the marginal utility $u'(\delta^1 + (P_j + y_j)\frac{V_X}{\eta^1_k} + b^1_k)$, which explains the MRS of $\frac{1}{\lambda^1}u'(\delta^1 + (P_j + y_j)\frac{V_X}{\eta^1_k} + b^1_k)$. Note that in spite of having a constant marginal utility, productive agents do not behave as if they were risk-neutral. Incomplete markets and the possibility of being unproductive and credit-constrained in the next period affect their pricing.

Regarding bonds, they may be purchased by productive type-1 or type-2 agents depending on the state of the world. Since our equilibrium features security prices that depend solely on the current state of the world, there is one subset of states of the world, characterized by the index subset I_1, in which only type-1 agents hold bonds, while type-2 agents do not. In states of the world I_1: (i) the Euler equation (24) of type-2 agents does not hold and (ii) the bond supply equals the demand of type-1 agents in equation (25). By the same token, there are states of the world, characterized by the subset index I_2, in which only type-2 agents buy bonds, while type-1 agents only buy stocks. This corresponds to the Euler equation (23) and the resource equality (26). Subsets I_1 and I_2 are potentially empty. Finally, in remaining states of world, i.e., those belonging to $\{1, \ldots, n\} - I_1 - I_2$, both type-1 and type-2 agents purchase bonds. The interpretation of equations (23)–(24) is the same as for (22)

Note that we need to explicitly check that the equilibrium exists, and that the consumption levels implied by equations (22)–(27) are consistent with the equilibrium structure. For example, we need to verify that unproductive agents are actually credit-constrained at the equilibrium. These conditions, discussed in Section 4 of the Appendix, implicitly define the set $\mathcal{V}$ of admissible security supplies V_X and V_B.

3.3 Discussion of our assumptions

As explained in the discussion of Proposition 1, our equilibrium relies on two assumptions: (i) the linear sections of the utility function and (ii) the upper bound on security volumes. The concave-linear utility function generalizes Fishburn's (1977) contribution. The shape of the period utility function implies that agents with a low consumption level are sensitive to small variations in consumption levels, while agents consuming a higher amount (i.e., those in a linear part) have a marginal utility that is invariant to small changes in consumption. However, these agents are not risk neutral since they can experience a significant increase in marginal utility if they are hit by a negative idiosyncratic shock that forces them to consume a low amount (which would be valued by the strictly concave part of the utility function). The concave-linear utility function accounts for extensive variations in consumption due to individual shocks but neglects the impact of small intensive variations. We consider this a simplified but relevant representation of consumption smoothing and of behavior with respect to idiosyncratic risk.

The upper bound on security volumes is the second important assumption. For our limited-heterogeneity equilibrium to exist, we have to limit the amount of self-insurance, such that unproductive agents remain credit-constrained. This assumption is not unrealistic for the bottom 50% of US households. For the top 50%, we justify our assumption by the fact that the assets considered here represent only a small share of the total amount of assets observed in the data. Not all assets are available for self-insurance, either because they are not liquid (such as stocks locked in retirement plans – see Kaplan and Violante 2014a and Kaplan, Violante, and Weidner 2014b) or because households do not wish to trade them (due to the so-called portfolio inertia reported in Brunnermeier and Nagel 2008 and in Bilias, Georgarakos, and Haliassos 2010). The modeling strategy is thus relevant for agents who have relatively few assets to self-insure, or if one assumes that the main tools available for self-insurance are captured in the income process (as a reduced form). Our small-trade equilibrium therefore enables us to analyze the effect of *additional* liquidity.

Other assumptions are much less critical with respect to the existence of our equilibrium. For instance, Assumption A could be replaced by a less strict assumption, at the cost of a greater number of agent classes in the equilibrium. As discussed in Footnote 9, we could also allow agents to hold negative wealth, as long as the borrowing limit is not too loose. Our equilibrium would still exist, provided that the set $\mathcal{V}$ of admissible volumes in Proposition 1 is changed accordingly.[17]

[17] There is a degree of substitution between the negative wealth constraint and the maximal bounds

4 Quantitative exercise

We now illustrate our model mechanisms in a quantitative exercise, using a plausible calibration for model parameters.

Assumption B implies that only type-1 agents hold stocks, while type-2 do not. This is consistent with the empirical observation that only 50% of US households hold stocks either directly or indirectly and that stock-holders are mostly in the top 50% of the income distribution (Bricker et al. 2014). We identify type-1 participating households as being in the top 50% of US households in terms of income distribution (henceforth the *top 50%*) and type-2 non-participating agents as being in the bottom 50% (henceforth the *bottom 50%*).

Model parameters are divided into two groups. First, we calibrate certain parameters to standard values (that ensure the existence of an equilibrium). Second, we use the tractability of our framework to simulate the model to match several targets (described below).

4.1 Parameter restrictions

4.1.1 Aggregate risk and asset volumes

The period is a quarter. There are two aggregate states ($n = 2$), which can be either G (for good) – corresponding to a boom – or B (for bad) – corresponding to a recession. For transition probabilities, we rely on Hamilton's (1994) estimation for recessions and booms and choose $\pi_{GG} = 0.75$ and $\pi_{BB} = 0.5$. The good state is thus more persistent than the bad one.

The volumes of assets are chosen to ensure that our equilibrium exists and we set $V_X = 0.002$ for stocks and $V_B = 0.1$ for bonds.

(in $\mathcal{V}$), permitting the existence of an equilibrium.

4.1.2 Preference parameters

The shape of the period utility function $\tilde{u}$ is defined by three parameters – see (3). First, we set $\tilde{u}(c) = \log(c)$ in the concave part. Second, to avoid arbitrariness in the choice of the slopes $\lambda^1 < \lambda^2$ of the two linear parts, we require them to be equal to the derivatives – computed at the relevant point – of the utility function $\log(c)$:

$$\lambda^i = \frac{1}{c_{GG}^{i,pp}},\ i = 1, 2, \tag{28}$$

where $c_{GG}^{i,pp}$ is the consumption level of type-$i = 1, 2$ agents, who have been productive for at least two consecutive periods, while the aggregate state is G and was G in the previous period. Our choice for λ^1 and λ^2 is consistent with our interpretation of the linear parts in the utility function as an approximation of a more general utility function. Obviously, the values of consumption level $c_{GG}^{i,pp}$ in turn depend on λ^i, which implies that equation (28) defining λ^i in fact involves solving a fixed-point problem.

4.1.3 Parameter restrictions

To bring discipline to the calibration strategy, we impose some further constraints on the model parameters, which are consistent with the mechanisms identified in Section 3.1. We first set $\omega_B^1 = 1.0$ to scale the income process of type-1 agents. Second, we set $\eta_k^1 = \eta_k^2 = \eta$ ($k = G, B$) such that a constant fraction of the population is productive in every period. Third, we assume that the income risk faced by type-2 agents is not time-varying, such that $\omega^2 \equiv \omega_G^2 = \omega_B^2$. Finally, we assume that the average value of dividends is $\overline{y} = 1$, such that only the ratio y_G/y_B is used in the calibration strategy.

4.2 Calibration

We are left with 11 parameters to calibrate: the discount factor β, the dividend process y_G/y_B, and 9 parameters driving the income process: 4 probabilities α_k^i ($i = 1, 2$ and $k = G, B$), 4 income levels ω_G^1, ω^2, δ^1, δ^2, and the share η of productive agents. Using equation (1), the values of $\rho^i_{k_1k_2}$ ($i = 1, 2$ and $k_1, k_2 = G, B$) are uniquely determined by the values of η and α^i_k for $i = 1, 2$ and $k = G, B$. To calibrate these 11 parameters, we match 12 empirical targets, which we denote by $T = [T_1, \ldots, T_{12}]$.

4.2.1 Consumption and household risk exposure

We first target parameters concerning consumption risk. In line with the literature (Parker and Vissing-Jorgensen, 2009, among others), the risk faced by each category of household is proxied by the volatility of the consumption growth rate for non-durable goods and services. Consumption is measured by quarterly expenditures on non-durable goods and on a subset of services deflated with the relevant price index. We use data from the Consumer Expenditure Survey (CEX) from 1980 to 2007.[18] A detailed discussion can be found in Appendix F. Our first two targets are the standard deviations of the quarterly consumption growth rate for the top 50%, equal to $T_1 \equiv \sigma(\Delta \log C_1) = 0.14$, and for the bottom 50%, equal to $T_2 \equiv \sigma(\Delta \log C_2) = 0.19$. The bottom 50% face higher total risk than the top 50%, as is standard in the literature. Our third target is the standard deviation of aggregate consumption, which is $T_3 \equiv \sigma(\Delta \log C^{tot}) = 1\%$. This last value is not implied by the first two targets, because of agent heterogeneity.

We also target the exposure of both groups to aggregate shocks. Following Parker and Vissing-Jorgensen (2009), we compute, for each group, the coefficient equal to: (Change in real group consumption per household)∗(Group share of population)/(Lagged aggregate real consumption per household).[19] The coefficients, which sum to one for both groups, can be interpreted as the fraction of aggregate risk born by each group. According to this metric, the top 50% bear $T_4 \equiv S_1 = 84\%$ of aggregate risk, whereas the bottom 50% bear the remaining 16%. Finally, the consumption share of the top 50% amounts to

[18] In what follows, we apply the methodology of Parker and Vissing-Jorgensen (2009) to a different subset of households in order to be consistent with our model. It is generally accepted that consumption data are not as accurate as data on household income (see Aguiar and Bils, 2015, among others, for a discussion). Nevertheless, as our results are consistent with those derived using different datasets, we are confident that the facts presented here are robust.

[19] This measure does not depend on the share of type-1 agent consumption in total consumption, as explained by Parker and Vissing-Jorgensen (2009).

$T_5 \equiv C_1/C_{tot} = 72.1\%$, which drives consumption inequalities in our economy.

4.2.2 Asset returns and the dividend process

We target four moments for stock and bond returns, and three moments for the dividend process. These seven moments are taken from Campbell's (1999) dataset. First, the stock returns are computed from the S&P 500 index, while the bond returns are computed from the six-month commercial paper rate. For ease of comparison, these returns are annualized quarterly real returns and correspond to historical US data from 1890–1991. The average bond interest rate is $T_6 \equiv E(R_f) = 0.9\%$, while the standard deviation of the bond interest rate is $T_7 \equiv \sigma(R_f) = 1.7\%$. The average stock return is equal to $T_8 \equiv E(R_s) = 8.1\%$, while its standard deviation is $T_9 \equiv \sigma(R_s) = 15.6\%$. The last three targets are the average price dividend ratio $T_{10} \equiv E(P_s/D) = 21$, the standard deviation of the log of the price dividend ratio $T_{11} \equiv \sigma(\log(P_s/D)) = 30\%$, and the standard deviation of the log of dividend growth $T_{12} \equiv \sigma(\Delta \log(D)) = 28.3\%$.

4.2.3 Parameter values

Table 1 presents the model parameter values. First, we find that the income of type-1 agents barely moves, as the income in the good state $\omega_G^1 = 1.01$ is very close to the income in the bad state $\omega_B^1 = 1$. As a consequence, time variation in the income process is mostly driven by time-varying probabilities. The probabilities α_G^1, α_B^1, α_G^2, and α_B^2 oscillate around the value of 0.9, which is close to, but slightly lower than, the value obtained when identifying idiosyncratic risk with employment risk. In fact, using US data from 1948Q1-2007Q4, the quarterly probability of remaining employed equals 0.953, based on Shimer's (2005) methodology (see Challe and Ragot, 2014). The share of productive agents is $\eta = 0.89$. The discount factor β amounts to 0.86, which ensures the existence of an equilibrium (see condition (10)) and a realistic price-dividend ratio. This discount factor is lower than the typical value used in complete-market models. Incomplete insurance-market models are known to require a lower discount factor to reach the same average returns (for instance Krusell, Mukoyama, and Smith et al. 2011). The dividend process

$y_G/y_B = 1.12$ allows us to match the standard deviation of the dividend quarterly growth rate. Finally, the two preference parameters λ^1 and λ^2 are described by equation (28).

Calibrated parameters									***"Equilibrium" parameters***	
π_{GG}	π_{BB}	σ	ω_B^1	V_X	V_B				λ^1	λ^2
0.75	0.50	1	1	0.002	0.1				0.99	2.25
Other parameters										
α_G^1	α_B^1	α_G^2	α_B^2	ω_G^1	$\omega_G^2 = \omega_B^2$	δ^1	δ^2	η	β	y_G/y_B
0.91	0.94	0.90	0.88	1.01	0.45	0.32	0.07	0.89	0.86	1.12

Table 1: Parameter values

4.3 Results

Table 2 summarizes the targets and model outcome. Based on simplifying assumptions – such as the volume of securities – the model provides reasonable quantitative outcomes. Consumption growth volatilities are close to their empirical counterpart, as are financial returns. We find a low return for the safe asset and a high equity premium of 7%.

	US Data	**Model**	**Description and Remarks**
Consumption Growth			
$\sigma(\Delta \log C_1)$(in %)	14	15	std. dev. of agg. type 1 cons. growth
$\sigma(\Delta \log C_2)$(in %)	19	22	std. dev. of type 2 cons. growth
$\sigma(\Delta \log C^{tot})$(in %)	1.0	1.0	std. dev. of agg. cons. growth
S_1(in %)	84	85	share of agg. risk born by type 1
C_1/C_{tot}(in %)	72	70	cons. share of type 1 in agg. cons.
Asset Returns			
$E(R_f)$(in %)	0.9	0.9	average safe return
$\sigma(R_f)$(in %)	1.7	1.2	std. dev. of the safe return
$E(R_s)$(in %)	8.1	7.9	average risky return
$\sigma(R_s)$(in %)	15.6	6.8	std. dev. of the risky return
Price-Dividend (P/D) Ratio			
$E(P_s/D)$	21	13	average P/D ratio
$\sigma(\log(P_s/D))$(in %)	30	17	std. dev. of log of P/D ratio
$\sigma(\Delta\log(D))$(in %)	28.3	27.7	std. dev. of log dividend growth

Note: See the text for a description of the statistics.

Table 2: Targets and model outcomes

The model also generates average moments of asset returns that are broadly in line with the data, although the risky return is more volatile than in the data. Finally, we find that the average income of type-1 agents is roughly twice as high as for type-2 agents. These values are consistent with empirical estimates of the skill premium, which is between 1.4 and 1.7 (Murphy and Welch, 1992) and with the fact that populations of skilled and unskilled workers have remained roughly the same size over the last 20 years (Mukoyama and Sahin, 2006).

Implicit valuation of the risky asset by type-2 agents. In Assumption B, we set a high participation cost for type-2 agents to ensure that they did not trade any stock. In equation (29), we provide a value of $\overline{\chi}_2$ to ensure that holding stocks is the dominated choice. Based on our calibration and the median annual income of US households of $52,250 in 2014, we find an annual participation cost of $33. This is relatively small compared to other participation cost estimates (Vissing-Jorgensen, 2002, among others).[20]

Role of participation costs. The ability of the previous model to reproduce asset prices crucially relies on limited participation in the stock market. To examine this, we perform the same quantitative analysis as in the previous section, however we relax Assumption B and set all participation costs to zero: $\chi_1 = \chi_2 = 0$. As explained in Proposition 4, both types of agents may trade stocks and bonds. We provide the full model in Appendix H. We re-calibrate the model without participation costs and find that the safe and risky returns are almost identical and equal to 7.0%. The model without participation costs thus fails to reproduce realistic asset prices.[21]

[20] As this calibration may depend on the small volume of debt, we also compute the implicit valuation of the risky asset by type-2 agents (i.e., their valuation with their own pricing kernel). We find that type-2 agents will never participate in the stock market, even if they face a proportional participation cost as low as 1.1%.

[21] The role of participation costs in asset prices confirms the findings of Guvenen (2009) in a model with heterogeneous preferences. Krusell, Mukoyama, and Smith (2011) have shown that in such an economy, it is not possible to reproduce empirical asset prices with realistic idiosyncratic risks and a low risk aversion. Participation cost is thus a key ingredient of our model's ability to match empirical data.

5 Conclusion

We have constructed an analytically tractable incomplete insurance-market model with limited participation in financial markets, heterogeneity in risk exposure, and aggregate shocks. Our small-trade equilibrium relies on not-too-large security volumes and a concave-linear utility function, as introduced by Fishburn (1977). Although simple, this model satisfactorily reproduces asset price properties and household risk exposure. This parsimonious setup could be used to study other forms of heterogeneity with aggregate shocks and, for instance, the heterogeneity of agents in terms of the two sides of their balance sheet (i.e., asset and liability sides). This would allow us to study financial intermediation in an incomplete market setting with aggregate shocks. Such an environment could improve our understanding of the way that financial markets function.

Appendix

A Participation costs

We now derive explicitly the condition on the participation cost $\overline{\chi}_2$ of Assumption B for type-2 agents to never participate in the stock market. This cost is such that participating in the stock market is a dominated strategy, no matter the state of the world.

If type-2 agents participate in stock markets, their portfolio choice is denoted $\{\tilde{x}_k^2, \tilde{b}_k^2\}_{k=1,\dots,n}$ given equilibrium prices $(P_k, Q_k)_{k=1,\dots,n}$. Purchasing the stock quantity $\tilde{x}_k^2$ is a dominated strategy in any state k, if investing the same amount in bonds offers in every state a greater payoff. Due to participation cost, purchasing $\tilde{x}_k^2$ costs $P_k\tilde{x}_k^2 + \chi_2$ and pays off $\tilde{x}_k^2(P_j + y_j)$ in the next period when the state of the world is $j = 1, \dots, n$. Investing the same amount $P_k\tilde{x}_k^2 + \overline{\chi}_2$ in bonds pays off $\frac{P_k\tilde{x}_k^2+\overline{\chi}_2}{Q_k}$ units of consumption in all states of the next period. In consequence, if $\frac{P_k\tilde{x}_k^2+\chi_2}{Q_k} > \tilde{x}_k^2(P_j + y_j)$ for any k, j, type-2 agents never wish to trade stocks. We deduce the following expression for $\overline{\chi}_2$ that ensures Assumption B to hold:

$$\overline{\chi}_2 = \max_{k,j=1,\dots,n} (Q_k(P_j + y_j) - P_k)\tilde{x}_k^2. \tag{29}$$

Following the same steps as in Proposition 4, we obtain for $\{\tilde{x}_k^2, \tilde{b}_k^2\}_{k=1,\dots,n}$:[22]

$$P_k \geq \beta \sum_{j=1}^{n} \pi_{kj}(\alpha_k^2 + (1-\alpha_k^2)\frac{1}{\lambda^2}u'(\delta^2 + (P_j + y_j)\tilde{x}_k^2 + \tilde{b}_k^2))(P_j + y_j),\ k \in \{1, \dots, n\}, \tag{30}$$

$$Q_k \geq \beta \sum_{j=1}^{n} \pi_{kj}(\alpha_k^2 + (1-\alpha_k^2)\frac{1}{\lambda^2}u'(\delta^2 + (P_j + y_j)\tilde{x}_k^2 + \tilde{b}_k^2)),\ k \in \{1, \dots, n\} - I_2, \tag{31}$$

where (30) and (31) hold with equality if $\tilde{x}_k^2 > 0$ and $\tilde{b}_k^2 > 0$ respectively.

B Proof of Proposition 1

We prove that the market arrangement implied by Assumption B, in which type-1 agents trade stocks, while type-2 do not, is an equilibrium. We proceed in two steps: (i) we prove that we can find prices and quantities such that equations (22)–(27) hold and (ii) we check that unproductive agents do not participate in security markets and that equilibrium consumption levels lie in proper definition sets of the utility function.

[22]We have used the fact that credit constraints bind for unproductive type-2 agents after this deviation, which is true in the equilibrium under consideration.

First step: the existence of prices and quantities. We define a correspondence on a compact set to invoke the Kakutani's fixed-point theorem. First, we define the compact convex sets $D_b = \{b \in \mathbb{R}: \ V_B \geq b \geq 0\}$ and $D_p = \{(P,Q) \in \mathbb{R}^2 : \underline{P} \leq P \leq \overline{P}$ and $0 \leq Q \leq \overline{Q}\}$ with $\underline{P} = \frac{\beta \min_{z\in Z} \alpha^1(z) y(z)}{1-\beta \min_{z\in Z} \beta\alpha^1(z)} > 0$, $\overline{P} = \frac{\beta \max_{z\in Z}(\alpha^1(z)+(1-\alpha^1(z))\frac{1}{\lambda^1}u'(\delta^1))y(z)}{1-\beta\max_{z\in Z}\alpha^1(z)+(1-\alpha^1(z))\frac{1}{\lambda^1}u'(\delta^1)} \geq \underline{P}$ and $\overline{Q} = \max_{z\in Z}\beta(\alpha^2(z)+(1-\alpha^2(z))\frac{1}{\lambda^2}u'(\delta^2))$.[23]

We define the following correspondence $\psi^p : \mathcal{F}(Z, D_b) \rightrightarrows \mathcal{P}\left(\mathcal{F}(Z, D_p)\right)$, as:[24]

$$
\begin{aligned}
\psi^p(b) = \{&(P,Q) \in \mathcal{F}(Z, D_p)| \\
&P = \beta E^z[(\alpha^1(z) + (1-\alpha^1(z))\frac{1}{\lambda^1}u'(\delta^1 + (P(z')+y(z'))\frac{V_X}{\eta^1(z)} + \frac{V_B - b}{\eta^1(z)}))(P(z')+y(z'))], \\
&\left. Q = \beta E^z[\alpha^2(z) + (1-\alpha^2(z))\frac{1}{\lambda^2}u'(\delta^2 + \frac{b}{\eta^2(z)})]\right\},
\end{aligned}
$$

where $E^z[\zeta] = \sum_{z'\in Z}\pi_{zz'}\zeta_{z'}$ is the conditional expectation of ζ.

If security demands solely depend on the current aggregate and idiosyncratic states, we deduce from Assumption A that the bond market clearing implies that $\forall z \in Z,\ b^2(z) = \frac{b(z)}{\eta^2(z)}$ and $b^1(z) = \frac{V_B - b(z)}{\eta^1(z)}$ where b^i denotes the bond demand of a type-i agent $(i = 1, 2)$.

We introduce the correspondence $\psi^x : \mathcal{F}(Z, D_p) \rightrightarrows \mathcal{P}\left(\mathcal{F}(Z, D_b)\right)$, as follows:

$$
\psi^x(P,Q) = \left\{b \in \mathcal{F}(Z, D_b) | T^p_{P,Q}(b) = 0, V_B \geq b(z) \geq 0\right\} \tag{32}
$$

where $\forall (P,Q) \in \mathcal{F}(Z, D_p),\ T^p_{P,Q}: \ b \in \mathcal{F}(Z, D_b) \mapsto \forall z \in Z,$

$$
\begin{aligned}
&(V_B - b(z)) \times 1_{E^z[\alpha^2(z)+(1-\alpha^2(z))\frac{1}{\lambda^2}u'(\delta^2+\frac{V_B}{\eta^2(z)})] > E^z[\alpha^1(z)+(1-\alpha^1(z))\frac{1}{\lambda^1}u'(\delta^1+(P+y(z'))\frac{V_X}{\eta^1(z)})]} \\
&+ b(z) \times 1_{E^z[\alpha^2(z)+(1-\alpha^2(z))\frac{1}{\lambda^2}u'(\delta^2)] < E^z[\alpha^1(z)+(1-\alpha^1(z))\frac{1}{\lambda^1}u'(\delta^1+(P+y(z'))\frac{V_X}{\eta^1(z)}+\frac{V_B}{\eta^1(z)})]} \\
&+ (E^z[\alpha^2(z) + (1-\alpha^2(z))\frac{u'(\delta^2 + \frac{b(z)}{\eta^2(z)})}{\lambda^2}] \\
&\qquad\qquad - E^z[\alpha^1(z) + (1-\alpha^1(z))\frac{u'(\delta^1 + (P+y(z'))\frac{V_X}{\eta^1(z)} + \frac{V_B - b}{\eta^1(z)})}{\lambda^1}]) \\
&\times 1_{E^z[\alpha^2(z)+(1-\alpha^2(z))\frac{1}{\lambda^2}u'(\delta^2+\frac{V_B}{\eta^2(z)})] \leq E^z[\alpha^1(z)+(1-\alpha^1(z))\frac{1}{\lambda^1}u'(\delta^1+(P+y(z'))\frac{V_X}{\eta^1(z)})]} \\
&\times 1_{E^z[\alpha^2(z)+(1-\alpha^2(z))\frac{1}{\lambda^2}u'(\delta^2)] \geq E^z[\alpha^1(z)+(1-\alpha^1(z))\frac{1}{\lambda^1}u'(\delta^1+(P+y(z'))\frac{V_X}{\eta^1(z)}+\frac{V_B}{\eta^1(z)})]},
\end{aligned}
$$

where $1_A = 1$ if A is true and 0 otherwise. The mapping $T^p_{P,Q}$ considers the three possible

[23] It will be straightforward to check that equilibrium prices and quantities respectively belong to D_p and D_b.

[24] Correspondences are set-valued functions (see Mas-Collel, Whinston and Green(1995), Section M.H). $\mathcal{P}(\star)$ is the set of all subsets of $\star$. For any compact K, $\mathcal{F}(Z,K)$ is the set of functions from Z to K and is isomorphic to K^n (and thus compact) since Z is of a cardinal n.

cases of bond market participation. Bonds are traded by: (i) only type-2 agents, (ii) only type-1 agents and (iii) both agents. These three cases correspond to three mutually exclusive conditions. We can therefore check that ψ^x is compact- and convex-valued and upper semi-continuous (since it is compact-valued and its graph is closed).[25] ψ^x is also non-empty: either there is complete market separation (with only type-1 or type-2 agents holding bonds), or both types of agents trade bonds.

Regarding ψ^p, we can also check that ψ^p is compact- and convex-valued and upper semi-continuous. We need to prove that ψ^p is not empty. Define the mapping T^x from $(b\mathcal{C}(Z \times D_b), \|\cdot\|_\infty)$ onto itself as follows:[26]

$$T^x : X \mapsto y(z) + \beta E^z \left[(\alpha^1(z) + (1-\alpha^1(z))\frac{1}{\lambda^1} u'(\delta^1 + X(z',\cdot)\frac{V_X}{\eta^1(z)} + \frac{V_B - b}{\eta^1(z)}))X(z',\cdot) \right],$$

where we can find $\overline{\beta}$ such that for all $z \in Z, X \in b\mathcal{C}(Z \times D_b)$, $0 < \beta(\alpha^1(z) + (1-\alpha^1(z))\frac{1}{\lambda^1}u'(\delta^1 + X\frac{V_X}{\eta^1(z)} + \frac{V_B-b}{\eta^1(z)}))) \leq \overline{\beta} < 1$ (condition (10)). We wish to prove that T^x is a contraction. For $X, X' \in b\mathcal{C}(Z \times D_b)$, $T^x X - T^x X'$ becomes after some manipulation:

$$\begin{aligned} T^x X - T^x X' &= \beta E^z \left[(\alpha^1(z) + (1-\alpha^1(z))\frac{1}{\lambda^1} u'(\delta^1 + X\frac{V_X}{\eta^1(z)} + \frac{V_B - b}{\eta^1(z)}))\,(X - X') \right] \\ &+ \beta E^z \left[(1-\alpha^1(z))\frac{1}{\lambda^1} \left(u'(\delta^1 + X\frac{V_X}{\eta^1(z)} + \frac{V_B - b}{\eta^1(z)})) - u'(\delta^1 + X'\frac{V_X}{\eta^1(z)} + \frac{V_B - b}{\eta^1(z)})) \right) X' \right]. \end{aligned}$$

We want to bound $\|T^x X - T^x X'\|_\infty$. The first term can be bounded by $\overline{\beta}\,\|X - X'\|_\infty$. The second term is null when $V_X = 0$. Let

$$\begin{aligned} \mathcal{V}_T = \Bigg\{ (V_X, V_B) \in (\mathbb{R}^+)^2 | \sup_{b \in D_b, X, X' \in b\mathcal{C}(Z \times D_b)} \beta(1-\alpha^1(z))\frac{1}{\lambda^1} \\ \left| u'(\delta^1 + X\frac{V_X}{\eta^1(z)} + \frac{V_B - b}{\eta^1(z)})) - u'(\delta^1 + X'\frac{V_X}{\eta^1(z)} + \frac{V_B - b}{\eta^1(z)})) \right| |X'| < (1-\overline{\beta})\,\|X - X'\|_\infty \Bigg\}. \end{aligned}$$

Note that $\mathcal{V}_T$ is not empty since it contains at least $(0,0)$ and, by continuity of u'', contains an open set containing $(0,0)$. By construction, for $(V_X, V_B) \in \mathcal{V}_T$, we have $\|T^x X - T^x X'\|_\infty < \|X - X'\|_\infty$. Edelstein's fixed-point theorem implies then that T^x

[25] Considering $\phi : p \mapsto \left\{ x \in [\underline{x}, \overline{x}], (\overline{x} - x)1_{p > k_2} + (x - \frac{k_2 - p}{k_2 - k_1}\underline{x} - \frac{p - k_1}{k_2 - k_1}\overline{x})1_{k_2 \geq p \geq k_1} + (x - \underline{x})\ 1_{p < k_1} = 0 \right\}$ $(k_2 > k_1)$ may clarify this point. $\phi(p) = \{\overline{x}\}$ for $p > k_2$; $\phi(p) = \{\frac{k_2 - p}{k_2 - k_1}\underline{x} + \frac{p - k_1}{k_2 - k_1}\overline{x}\}$ for $k_2 \geq p \geq k_1$ and $\phi(p) = \{\underline{x}\}$ for $p < k_2$. The set $\{(p, \phi(p))\,,\ p \in \mathbb{R}\}$ is closed.

[26] $b\mathcal{C}(\star)$ is the set of continuous bounded functions over the metric space $\star$, endowed with the sup. norm.

admits a unique $X \in b\mathcal{C}(Z \times D_b)$ such that:

$$X(z,b) = y + \beta E^z \left[(\alpha^1(z) + (1-\alpha^1(z))\frac{1}{\lambda^1}u'(\delta^1 + X(z',\cdot)\frac{V_X}{\eta^1(z)} + \frac{V_B - b}{\eta^1(z)}))X(z',\cdot) \right].$$

We have just proven that the stock price $P(\cdot) = X(\cdot) - y$ is well-defined and is a continuous function of bond demand. We deduce that ψ^p is not empty.

We finally define the correspondence $\psi : ((P,Q),b) \in \mathcal{F}(Y,D_p)\times\mathcal{F}(Y,D_b) \rightrightarrows (\psi^p(b), \psi^x(P,Q)) \in \mathcal{P}(\mathcal{F}(Y,D_p) \times \mathcal{F}(Y,D_b))$. Since ψ^p and ψ^x are non-empty, compact- and convex-valued and upper semi-continuous, ψ also is. The Kakutani's theorem then ensures the existence of a fixed point $((P^*,Q^*),b^*) \in (\psi^p(b^*), \psi^x(P^*,Q^*))$. It is then straightforward to check that this fixed-point defines a competitive equilibrium.

We now check that unproductive agents are kept out of the financial market.

Second step: unproductive agents do not participate in security markets. First note that the fixed-point generates an equilibrium with endogenous bond market participation of productive type-1 and type-2 agents. However, we need to determine, under which conditions unproductive agents of both types choose not to trade any security.

Security zero-supplies. We first assume $V_X = V_B = 0$. No security is traded and security prices are given by:

$$P(z) = \beta(\alpha^1(z) + (1-\alpha^1(z))\frac{1}{\lambda^1}u'(\delta^1))E^z\left[(P(z') + y(z'))\right], \tag{33}$$

$$Q(z) = \beta(\alpha^2(z) + (1-\alpha^2(z))\frac{1}{\lambda^2}u'(\delta^2)). \tag{34}$$

The equilibrium existence conditions are as follows (here $\hat{z}$ is the former state, z the current one and z' the next one):

$$P(z)\frac{1}{\lambda^1}u'(\delta^1) > \beta E^z \left[(1-\rho^1(z,z') + \rho^1(z,z')\frac{1}{\lambda^1}u'(\delta^1))(P(z') + y(z')) \right], \tag{35}$$

$$Q(z)\frac{1}{\lambda^1}u'(\delta^1) > \beta(1 - E^z\left[\rho^1(z,z')\right] + E^z\left[\rho^1(z,z')\right]\frac{1}{\lambda^1}u'(\delta^1)), \tag{36}$$

$$Q(z)\frac{1}{\lambda^2}u'(\delta^2) > \beta(1 - E^z\left[\rho^2(z,z')\right] + E^z\left[\rho^2(z,z')\right]\frac{1}{\lambda^2}u'(\delta^2)). \tag{37}$$

First notice that condition (35) can be expressed using (33) as:

$$E^z\left[\left(\alpha^1(z) + (1-\alpha^1(z))\frac{1}{\lambda^1}u'(\delta^1))\frac{1}{\lambda^1}u'(\delta^1) - (1-\rho^1(z,z') + \rho^1(z,z')\frac{1}{\lambda^1}u'(\delta^1)) \right) (P(z') + y(z')) \right] > 0.$$

For conditions (35)–(37) to hold, it is sufficient that using (33)–(34), we have:

$$(\alpha^1(z) + (1-\alpha^1(z))\frac{1}{\lambda^1}u'(\delta^1))\frac{1}{\lambda^1}u'(\delta^1) > 1 - E^z\left[\rho^1(z,z')\right] + E^z\left[\rho^1(z,z')\right]\frac{1}{\lambda^1}u'(\delta^1), \quad (38)$$

$$(\alpha^2(z) + (1-\alpha^2(z))\frac{1}{\lambda^2}u'(\delta^2))\frac{1}{\lambda^1}u'(\delta^1) > 1 - E^z\left[\rho^1(z,z')\right] + E^z\left[\rho^1(z,z')\right]\frac{1}{\lambda^1}u'(\delta^1), \quad (39)$$

$$(\alpha^2(z) + (1-\alpha^2(z))\frac{1}{\lambda^2}u'(\delta^2))\frac{1}{\lambda^2}u'(\delta^2) > 1 - E^z\left[\rho^2(z,z')\right] + E^z\left[\rho^2(z,z')\right]\frac{1}{\lambda^2}u'(\delta^2). \quad (40)$$

We can check that equations (38) and (40) can be seen as positivity inequalities of polynomial functions in $\frac{1}{\lambda^1}u'(\delta^1)$ and $\frac{1}{\lambda^2}u'(\delta^2)$ respectively. Each polynomial function admits one negative root and another root equal to 1. Both polynomials are thus always positive since $\frac{1}{\lambda^1}u'(\delta^1) > 1$ and $\frac{1}{\lambda^2}u'(\delta^2) > 1$ (see Assumption C). Conditions (38) and (40) therefore always hold. The condition (39) can similarly be written as a positivity inequality of a polynomial function in $\frac{1}{\lambda^1}u'(\delta^1)$ and $\frac{1}{\lambda^2}u'(\delta^2)$, which is increasing in both arguments. We therefore deduce that: (i) when $\frac{1}{\lambda^1}u'(\delta^1) \geq \frac{1}{\lambda^2}u'(\delta^2)$, condition (39) holds whenever condition (40) does and (ii) when $\frac{1}{\lambda^1}u'(\delta^1) \leq \frac{1}{\lambda^2}u'(\delta^2)$, condition (39) holds whenever condition (38) does. In consequence, condition (39) always holds.

We finally check that consumptions of productive (resp. unproductive) agents lie in the linear (resp. concave) part of the utility function. Since our equilibrium features limited-heterogeneity, there are only 4 different agents classes per type, each of which depends on the current and past productive status. For instance, $c^{i,pu}_{\hat{z},z}$ is the consumption of type-i agents, who are currently unproductive (in state z) but were productive in the previous period (in state $\hat{z}$). The consumption levels of the different classes ($i = 1, 2$) are: $c^{i,pp}_{\hat{z},z} = c^{i,up}_{\hat{z},z} = \omega^i(z)$ and $c^{i,pu}_{\hat{z},z} = c^{i,uu}_{\hat{z},z} = \delta^i$. Assumption C readily implies that consumptions lie in the proper regions of the utility function.

The equilibrium always exists in zero volume.

Positive supply economy. We assume that $V_B, V_X > 0$. Security prices are:

$$P(z) = \beta E_z\Big[\Big(\alpha^1(z) + (1-\alpha^1(z))\frac{1}{\lambda^1}u'(\delta^1 + b^1(z) + \frac{V_X}{\eta^1(z)}(P(z') + y(z')))\Big)(P(z') + y(z'))\Big], \quad (41)$$

$$Q(z) = \beta E_z[\alpha^2(z) + (1-\alpha^2(z))\frac{1}{\lambda^2}u'(\delta^2 + b^2(z))], \quad (42)$$

where the quantities b^1 and b^2 are determined by three cases (see definition (32) of ψ^x):

- $b^1(z) = 0$ and $b^2(z) = \frac{V_B}{\eta^2(z)}$ if $E_z[\alpha^2(z) + (1-\alpha^2(z))\frac{1}{\lambda^2}u'(\delta^2 + \frac{V_B}{\eta^2(z)})] \geq E_z[\alpha^1(z) + (1-\alpha^1(z))\frac{1}{\lambda^1}u'(\delta^1 + \frac{V_X}{\eta^1(z)}(P(z') + y(z')))]$: complete market segmentation;

- $b^1(z) = \frac{V_B}{\eta^1(z)}$ and $b^2(z) = 0$ if $E_z[\alpha^2(z) + (1-\alpha^2(z))\frac{1}{\lambda^2}u'(\delta^2)] \leq E_z[\alpha^1(z) + (1 -$

$\alpha^1(z))\frac{1}{\lambda^1}u'(\delta^1 + \frac{V_X}{\eta^1(z)}(P(z')+y(z')) + \frac{V_B}{\eta^1(z)})]$: also complete market segmentation;

- $b^1(z) = \frac{V_B - \eta^2 b^2(z)}{\eta^1(z)}$ and $b^2(z)$ solves $E_z[\alpha^2(z)+(1-\alpha^2(z))\frac{1}{\lambda^2}u'(\delta^2+b^2(z))] = E_z[\alpha^1(z)+(1-\alpha^1(z))\frac{1}{\lambda^1}u'(\delta^1 + \frac{V_X}{\eta^1(z)}(P(z')+y(z')) + \frac{V_B-\eta^2(z)b^2(z)}{\eta^1(z)})]$: both types trade bonds.

Since prices and bond quantities depend on V_X and V_B (in addition to other model parameters), equilibrium existence conditions can be expressed as $\Theta(V_X, V_B) > 0$, where:

$$\Theta(V_X,V_B) = \begin{pmatrix} P(z)\frac{1}{\lambda^1}u'(\delta^1 + b^1(\hat{z}) + \frac{V_X}{\eta^1(\hat{z})}(P(z)+y(z)))\dots \\ \dots - \beta E_z\left[(1-\rho^1(z,z') + \rho^1(z,z')\frac{1}{\lambda^1}u'(\delta^1))(P(z')+y(z'))\right] \\ Q(z)\frac{1}{\lambda^1}u'(\delta^1 + b^1(\hat{z}) + \frac{V_X}{\eta^1(\hat{z})}(P(z)+y(z)))\dots \\ \dots - \beta(1 - E^z\left[\rho^1(z,z')\right] + E^z\left[\rho^1(z,z')\right]\frac{1}{\lambda^1}u'(\delta^1)) \\ Q(z)\frac{1}{\lambda^2}u'(\delta^2 + b^2(\hat{z}))\dots \\ \dots - \beta(1 - E^z\left[\rho^2(z,z')\right] + E^z\left[\rho^2(z,z')\right]\frac{1}{\lambda^2}u'(\delta^2)) \end{pmatrix}_{(\hat{z},z)\in Z^2} > 0. \tag{43}$$

Since the set Z is of cardinal n, $\Theta(V_X, V_B) \in \mathbb{R}^{3n^2}$. Note that $\Theta(V_X, V_B) > 0$ means that every component of $\Theta(V_X, V_B)$ is strictly positive. Let:

$$\mathcal{V}_\Lambda = \left\{(V_X, V_B) \in (\mathbb{R}^+)^2 | \Theta(V_X, V_B) > 0\right\}. \tag{44}$$

The zero supply part implies that $\mathcal{V}_\Lambda$ is not empty and, by continuity, includes an open set (of $(\mathbb{R}^+)^2$ endowed with the Euclidean norm) containing $(0,0)$. In other words, there exist $\overline{V}_X^\Lambda > 0$ and $\overline{V}_B^\Lambda > 0$, such that for all $0 \le V_X \le \overline{V}_X^\Lambda$ and $0 \le V_B \le \overline{V}_B^\Lambda$, $(V_X, V_B) \in \mathcal{V}_\Lambda$.

We now turn to the consumption expression. The consumption levels of the different classes ($i = 1,2$) can be expressed as follows

$$c^{i,pp}_{\hat{z},z} = \omega^i(z)(1-\tau(z)) + \left(P(z)(\frac{V_X}{\eta^1(\hat{z})} - \frac{V_X}{\eta^1(z)}) + y(z)\frac{V_X}{\eta^1(\hat{z})} - \chi^1\right)1_{i=1} \tag{45}$$
$$+ b^i(\hat{z}) - Q(z)b^i(z),$$

$$c^{i,up}_{\hat{z},z} = \omega^i(z)(1-\tau(z)) - \left(P(z)\frac{V_X}{\eta^1(z)} + \chi^1\right) 1_{i=1} - Q(z)b^i(z), \tag{46}$$

$$c^{i,pu}_{\hat{z},z} = \delta^i + (P(z)+y(z))\frac{V_X}{\eta^1(\hat{z})} 1_{i=1} + b^i(\hat{z}), \tag{47}$$

$$c^{i,uu}_{\hat{z},z} = \delta^i. \tag{48}$$

where taxes are given by $\tau(z) = \frac{(1-Q(z))V_B}{\omega^1(z)\eta^1(z)+\omega^2(z)\eta^2(z)}$. The vector of consumptions is denoted $C(V_X, V_B) = [c^{1,pp}_{\hat{z},z}, c^{1,up}_{\hat{z},z}, c^{2,pp}_{\hat{z},z}, c^{2,up}_{\hat{z},z}, c^{1,pu}_{\hat{z},z}, c^{1,uu}_{\hat{z},z}, c^{2,pu}_{\hat{z},z}, c^{2,uu}_{\hat{z},z}]_{(\hat{z},z)\in Z^2}$ and depends on V_B and

V_X. The space of admissible consumptions is $\Gamma = \left([c_4^*, c_5^*]^2 \times [c_2^*, c_3^*]^2 \times [0, c_1^*]^4\right)^{n^2}$. Let

$$\mathcal{V}_\Gamma = \left\{(V_X, V_B) \in (\mathbb{R}^+)^2 | C(V_X, V_B) \in \Gamma\right\}. \tag{49}$$

As for $\mathcal{V}_T$ and $\mathcal{V}_\Lambda$, we know, from the zero supply part, that $\mathcal{V}_\Gamma$ is not empty and by continuity that an open set containing $(0,0)$ is included in $\mathcal{V}_\Lambda$.

Finally, let define the set $\mathcal{V}_1$ (which is non-empty and includes an open set with $(0,0)$) of volumes for which the equilibrium, where only type-1 agents trade stocks, exists:

$$\mathcal{V}_1 = \mathcal{V}_T \cap \mathcal{V}_\Lambda \cap \mathcal{V}_\Gamma. \tag{50}$$

C Proof of Proposition 2

Since dividends are IID, stock prices are constant. Provided that condition (10) holds, the Euler equation for the stock implies:

$$P^{ZV} = \frac{\beta(\alpha^1 + (1-\alpha^1)\frac{u'(\delta^1)}{\lambda^1})}{1 - \beta(\alpha^1 + (1-\alpha^1)\frac{u'(\delta^1)}{\lambda^1})} E^{\tilde{z}}[y(\tilde{z})], \tag{51}$$

where $E^{\tilde{z}}[\cdot]$ is the expectation with respect to $\tilde{z}$. Type-2 agents are trading riskless bonds, while the bond price is too expensive for type-1 agents, i.e.:

$$Q^{ZV} = \beta\left(\alpha^2 + (1-\alpha^2)\frac{u'(\delta^2)}{\lambda^2}\right), \tag{52}$$

$$Q^{ZV} > \beta\left(\alpha^1 + (1-\alpha^1)\frac{u'(\delta^1)}{\lambda^1}\right), \tag{53}$$

where condition (53) holds thanks to condition (11). The zero supply economy therefore features full market segmentation, where type-1 agents hold stocks, while type-2 agents hold bonds. This equilibrium always exists from Proposition 1.

From price expressions (51) and (52), we deduce the equity premium of equation (18).

D Proof of Proposition 3

Because the dividend process is IID, stock and bond prices, as well as bond holdings, are constant. The Euler equations for both securities become:

$$P^{PV} = \beta E^{\tilde{z}} \left[\left(\alpha^1 + (1-\alpha^1) \frac{u'(\delta^1 + (P_{t+1} + y_{t+1}) \frac{V_X}{\eta^1} + b^1)}{\lambda^1} \right) (P^{PV} + y(\tilde{z})) \right],$$

$$Q^{PV} = \beta(\alpha^2 + (1-\alpha^2) \frac{u'(\delta^2 + b^2)}{\lambda^2}).$$

We solve for the price expression in the neighborhood of zero volumes. We assume that $0 < V_X \ll 1$ and $0 < V_B \ll 1$. Since bonds cannot be short-sold, we also have $0 \leq b^i \ll 1$. We obtain $P^{PV} \approx P^{ZV} + \pi_x V_X + \pi_b b^1$,[27] where P^{ZV} defined in equation (51) is the stock price in zero volume and where:

$$\pi_x (1 - \beta\psi^1) = \beta(1-\alpha^1) \frac{u''(\delta^1)}{\lambda^1} E^{\tilde{z}} \left[(P^{ZV} + y(\tilde{z}))^2 \right], \tag{54}$$

$$\pi_b (1 - \beta\psi^1) = \beta(1-\alpha^1) \frac{u''(\delta^1)}{\lambda^1} E^{\tilde{z}} \left[P^{ZV} + y(\tilde{z}) \right], \tag{55}$$

$$\text{with: } \psi^i = \alpha^i + (1-\alpha^i) \frac{1}{\lambda^i} u'(\delta^i), \; i = 1, 2. \tag{56}$$

For the bond, we obtain $Q^{PV} \approx Q^{ZV} + \beta(1-\alpha^2)\frac{u''(\delta^2)}{\lambda^2} b^2$ for type-2 agents, where Q^{ZV} defined in equation (52) is the bond price in zero volume. From these equations, one gets equation (19).

For type-1 agents, we have $Q^{PV} \gtrsim \beta\psi^1 + \beta(1-\alpha^1)\frac{u''(\delta^1)}{\lambda^1}(b^1 + E^{\tilde{z}}[P^{ZV} + y(\tilde{z})])$. If type-1 agents do not participate to the bond market, the previous inequality is strict and we have $b^1 = 0$ and $b^2 = \frac{V_B}{\eta^2}$. If type-1 agents trade bonds, the previous inequality is an equality and noticing that $b^1 = \frac{V_B}{\eta^1} - \frac{\eta^2}{\eta^1} b^2$, we deduce the bond expressions (20) and (21). Because of condition (11), type-2 agents cannot be credit-constrained. Otherwise, we would have $(1-\alpha^1)\frac{u''(\delta^1)}{\lambda^1}(\frac{V_B}{\eta^1} + \frac{V_X}{\eta^1} E^{\tilde{z}}[P^{ZV} + y(\tilde{z})]) > \psi^2 - \psi^1 > 0$, contradicting positive volumes. We derive then from bond and stock prices the equity premium in (18).

[27] The approximation sign $\approx$ refers to a first order development with respect to security volumes. It should be understood as $\ldots = \ldots + o(V_X, V_B)$. We assume that both volumes have the same

E Proof of Proposition 4

From Proposition 1, we deduce that two subsets $I_i \subset \{1, \ldots, n\}$ $(i = 1, 2)$ characterize the states in which only type-i agents trade bonds, such that the $4n$ variables $(b_k^1, b_k^2, P_k, Q_k)_{k=1,\ldots,n}$ characterizing the equilibrium are given by the $4n$ equations (22)–(27).

For the equilibrium to exist, we need to check two sets of conditions: (i) productive agents of a given type are excluded from bond markets; (ii) since unproductive agents of both types are permanently excluded from both financial markets.

In the states of the world I_2 where only type-2 agents trade bonds, type-1 agents are excluded due to too high bond prices and the following inequality has to hold:

$$Q_k > \beta \sum_{j=1}^{n} \pi_{kj} (\alpha_k^1 \lambda^1 + (1 - \alpha_k^1) \frac{1}{\lambda^1} u'(\delta^1 + (P_j + y_j) \frac{V_X}{\eta^1})), \text{ for } k \in I_2. \tag{57}$$

By the same token, for states of the world I_1, where only type-1 agents trade bonds:

$$Q_k > \beta (\alpha_k^2 \lambda^2 + (1 - \alpha_k^2) \frac{1}{\lambda^2} u'(\delta^2)), \ k \in I_1. \tag{58}$$

Type-1 (unproductive) agents are excluded from both stock and bond markets. The two following inequalities therefore need to hold for all $k, h = 1, \ldots, n$:

$$P_k \frac{1}{\lambda^1} u'(\delta^1 + b_h^1 + \frac{V_X}{\eta_h^1}(P_k + y_k)) > \beta \sum_{j=1}^{n} \pi_{kj} (1 - \rho_{kj}^1 + \rho_{kj}^1 \frac{1}{\lambda^1} u'(\delta^1))(P_j + y_j), \tag{59}$$

$$Q_k \frac{1}{\lambda^1} u'(\delta^1 + b_h^1 + \frac{V_X}{\eta_h^1}(P_k + y_k)) > \beta \sum_{j=1}^{n} \pi_{kj} (1 - \rho_{kj}^1 + \rho_{kj}^1 \frac{1}{\lambda^1} u'(\delta^1)). \tag{60}$$

Unproductive type-2 agents cannot participate to stock markets. For them to be excluded from bond markets, the following inequality needs to hold for all $k, h = 1, \ldots, n$:

$$Q_k \frac{1}{\lambda^2} u'(\delta^2 + b_h^2 + \frac{V_X}{\eta_h^2}(P_k + y_k)) > \beta \sum_{j=1}^{n} \pi_{kj} (1 - \rho_{kj}^2 + \rho_{kj}^2 \frac{1}{\lambda^2} u'(\delta^2)). \tag{61}$$

F Data Appendix

We consider the dataset used by Heathcote, Perri and Violante (2010). To measure the consumption of non-durable and services, we use the sum of expenditures on non-durable goods, including: the vehicle services and other vehicle expenses (insurance, maintenance, etc.), the housing services, the rent paid, other lodging expenses, household equipment and entertainment. These items are deflated using the CPI. This measure corresponds to the variable *ndpnd*0 in Heathcote et al. (2010). We use the weights given in the CEX to define in each quarter the bottom 50% and the top 50% of households in the consumption distribution.

To compute the volatility of consumption growth for a given group in each quarter, we use the variance of the consumption growth rate between quarter t and quarter $t+1$ among all households belonging to said group at date t (regardless the household's group in $t+1$). We then compute the average variance per group over the time period.

G Description of the calibration algorithm

We describe here the algorithm of Section 4 that we use to minimize the distance between the 6 moments generated by the model and their empirical counterparts and that allows us to calibrate our model through the simulated method of moments. We denote $\chi_v = [\alpha^1, \alpha^2, \omega^2, \delta^1, \delta^2, \omega_G^1] \in \mathbb{R}_+^6$ the vector of model parameters we have to compute. We start from an initial guess vector χ_v^0.

1. We compute the six moments $\widetilde{T}^0$ generated by the model when parameters are equal to χ_v^0. We compute the score $S^0 = (\widetilde{T}^0 - T)\Omega(\widetilde{T}^0 - T)'$, where $\Omega = I_{6\times 6}$ is the weight matrix and T is the vector of empirical moments we match.

2. We construct the hyper-cube of the $2^6 = 64$ neighbors of χ_v^0 by considering marginal increase or decrease in each parameter: $\chi_v^i = [\chi_{v,1} \pm \epsilon, \ldots, \chi_{v,6} \pm \epsilon]$ $(i = 1, \ldots, 64)$ where we set $\epsilon = 10^{-3}$.

3. For every vector χ_v^i, we compute the moments generated by the model $\widetilde{T}^i$ $(i = 1, \ldots, 64)$. We then also compute the related score $S^i = (\widetilde{T}^i - T)\Omega(\widetilde{T}^i - T)'$ for $i = 1, \ldots, 64$.

4. If $S^0 \leq S^i$ for all $i = 1, \ldots, 64$, we stop the algorithm and we have just found a minimum. We then set our model parameters equal to χ_v^0. If not, we start the algorithm in step 1 with the new initial value $\chi_v^0 = \chi_v^{i_{\min}}$, where $i_{\min} = \arg\min_i S_i$.

This algorithm generates a path in $\mathbb{R}_+^6$ converging towards a (local) minimum. We try different starting points χ_v^0 to find a global minimum.

H The model without participation costs

Following the same steps as in the paper, we deduce the structure of the model without participation costs. At the equilibrium, both agents types may trade or not bonds and stocks. In particular, stock holdings x_k^1 and x_k^2 in state k are determined by Euler equations. There exist sets $I_i^B, I_i^X \subset \{1,\ldots,n\}$ $(i = 1,2)$, such that the $6\times n$ variables $(b_k^1, b_k^2, x_k^1, x_k^2, P_k, Q_k)_{k=1,\ldots,n}$ defining the equilibrium are given by the following $6\times n$ equations $(i = 1,2)$:

$$P_k = \beta\sum_{j=1}^{n}\pi_{kj}(\alpha_k^i + (1-\alpha_k^i)\frac{1}{\lambda^i}u'(\delta^i + (P_j+y_j)x_k^i + b_k^i))(P_j+y_j),\ k\in\{1,\ldots,n\} - I_i^X, \tag{62}$$

$$Q_k = \beta\sum_{j=1}^{n}\pi_{kj}(\alpha_k^i + (1-\alpha_k^i)\frac{1}{\lambda^i}u'(\delta^i + (P_j+y_j)x_k^i + b_k^i)),\ k\in\{1,\ldots,n\} - I_i^B, \tag{63}$$

$$V_X = \eta_k^i\, x_k^i \text{ and } 0 = x_k^j,\ k\in I_i^X,\ i\neq j = 1,2, \tag{64}$$

$$V_X = \eta_k^1\, x_k^1 + \eta_k^2\, x_k^2,\ k\in\{1,\ldots,n\} - I_1^X - I_2^X, \tag{65}$$

$$V_B = \eta_k^i\, b_k^i \text{ and } 0 = b_k^j,\ k\in I_i^B,\ i\neq j = 1,2, \tag{66}$$

$$V_B = \eta_k^1\, b_k^1 + \eta_k^2\, b_k^2,\ k\in\{1,\ldots,n\} - I_1^B - I_2^B. \tag{67}$$

The set I_i^X $i = 1,2$ gathers states of the world, in which type-i agents do not trade stocks. The set I_i^B has the same meaning for bond market. The sets I_1^B, I_2^B on one side and I_1^X, I_2^X on the other side must be disjoint. This means that there should not exist a state of the world, in which no one is trading bond or stocks. Note that since our equilibrium features security prices that only depend on the current state of the world, the sets I_i^B, I_i^X are not time-dependent.

As in the core of the paper, several inequalities have to hold for the above equations to define a small-trade equilibrium. These inequalities guarantee that: (i) productive agents who do not trade a given security do not want to do so (i.e., this implies inequalities similar to (57)–(58)), and (ii) that unemployed agents do not want to trade (i.e., this implies inequalities similar to (59)–(61)). For the sake of conciseness, we do not report here these inequalities, which are rather straightforward to deduce from the previous equilibrium but rather lengthy to write down.

References

Aguiar, M., and M. Bils (2015): “Has Consumption Inequality Mirrored Income Inequality?,” *American Economic Review*, 105(9), 2725–2756.

Allen, F., and D. Gale (1994): “Limited Market Participation and Volatility of Asset Prices,” *American Economic Review*, 84(4), 933–955.

Basak, S., and D. Cuoco (1998): “An Equilibrium Model with Restricted Stock Market Participation,” *The Review of Financial Studies*, 11(2), 309–341.

Becker, R., and I. Zilcha (1997): “Stationary Ramsey Equilibria under Uncertainty,” *Journal of Economic Theory*, 75(1), 122–140.

Bilias, Y., D. Georgarakos, and M. Haliassos (2010): “Portfolio Inertia and Stock Market Fluctuations,” *Journal of Money, Credit and Banking*, 42(4), 715–742.

Bricker, J., L. Dettling, A. Henriques, J. Hsu, K. Moore, J. Sabelhaus, J. Thompson, and R. Windle (2014): “Changes in U.S. Family Finances from 2010 to 2013: Evidence from the Survey of Consumer Finances,” *Federal Reserve Bulletin*, 100(4).

Brunnermeier, M. K., and S. Nagel (2008): “Do Wealth Fluctuations Generate Time-Varying Risk Aversion? Micro-evidence on Individuals,” *American Economic Review*, 98(3), 713–736.

Campbell, J. Y. (1999): “Asset Prices, Consumption, and the Business Cycle,” in *Handbook of Macroeconomics*, ed. by J. B. Taylor, and M. Woodford, vol. 1, Part C, chap. 19, pp. 1231–1303. Elsevier.

Cao, D. (2016): “Existence of Generalized Recursive Equilibrium in Krusell and Smith (1998),” Working Paper, Georgetown University.

Challe, E., F. LeGrand, and X. Ragot (2013): “Incomplete Markets, Liquidation Risk and the Term Structure of Interest Rates,” *Journal of Economic Theory*, 148(6), 2483–2519.

Challe, E., and X. Ragot (2014): “Precautionary Saving over the Business Cycle,” *Economic Journal*, 126(590), 135–164.

Chatterjee, S., D. Corbae, M. Nakajima, and J.-V. Ríos-Rull (2007): “A Quantitative Theory of Unsecured Consumer Credit with Risk of Default,” *Econometrica*, 75(6), 1525–1589.

Cheridito, P., and J. Sagredo (2016a): “Comment on "Competitive Equilibria of Economies with a Continuum of Consumers and Aggregate Shocks" by J. Miao,” Working Paper, ETH Zurich.

——— (2016b): “Existence of Sequential Competitive Equilibrium in Krusell-Smith Type Economies,” Working Paper, ETH Zurich.

Chien, Y., H. Cole, and H. Lustig (2011): “A Multiplier Approach to Understanding the Macro Implications of Household Finance,” *Review of Economic Studies*, 78(1), 199–234.

——— (2012): “Is the Volatility of the Market Price of Risk Due to Intermittent Portfolio Rebalancing?,” *American Economic Review*, 102(6), 2859–96.

Coleman, W. J. I. (1991): “Equilibrium in a Production Economy with an Income Tax,” *Econometrica*, 59(4), 1091–1104.

Constantinides, G. M., and D. Duffie (1996): “Asset Pricing with Heterogeneous Consumers,” *Journal of Political Economy*, 104(2), 219–240.

Constantinides, G. M., and A. Ghosh (2017): “Asset Pricing with Countercyclical Household Consumption Risk,” *Journal of Finance*, 72(1).

Dang, T. V., G. Gorton, B. Holmstrom, and G. Ordoñez (2017): “Banks as Secret

Keepers," *American Economic Review*, 107(4), 1005–1029.

DeGiorgi, G., and L. Gambetti (2017): "Business Cycle Fluctuations and the Distribution of Consumption," *Review of Economic Dynamics*, 23, 19–41.

Duffie, D., J. Geanakoplos, A. Mas-Colell, and A. McLennan (1994): "Stationary Markov Equilibria," *Econometrica*, 62(4), 745–781.

Favilukis, J. (2013): "Inequality, Stock Market Participation, and the Equity Premium," *Journal of Financial Economics*, 107(3), 740–759.

Feldman, M., and C. Gilles (1985): "An Expository Note on Individual Risk without Aggregate Uncertainty," *Journal of Economic Theory*, 35(1), 26–32.

Fishburn, P. (1977): "Mean-Risk Analysis with Risk Associated with Below-Target Returns," *American Economic Review*, 67(2), 116–126.

Gârleanu, N., L. Kogan, and S. Panageas (2012): "Displacement Risk and Asset Returns," *Journal of Financial Economics*, 105(3), 491–510.

Gomes, F. J., and A. Michaelides (2008): "Asset Pricing with Limited Risk Sharing and Heterogeneous Agents," *The Review of Financial Studies*, 21(1), 415–448.

Green, E. (1994): "Individual-Level Randomness in a Nonatomic Population," Working paper, University of Minnesota.

Guvenen, F. (2009): "A Parsimonious Macroeconomic Model for Asset Pricing," *Econometrica*, 77(6), 1711–1750.

Hall, R. (2010): *Forward-Looking Decision Making*. Princeton University Press.

Hamilton, J. D. (1994): *Time Series Analysis*. Princeton University Press.

Heathcote, J., F. Perri, and G. L. Violante (2010): "Unequal We Stand: An Empirical Analysis of Economic Inequality in the United States, 1967-2006," *Review of Economic Dynamics*, 13(1), 519–548.

Heathcote, J., K. Storesletten, and G. L. Violante (2009): "Quantitative Macroeconomics with Heterogeneous Households," *Annual Review of Economics*, 1(1), 319–354.

Heaton, J., and D. Lucas (1999): "Stock Prices and Fundamentals," in *NBER Macroeconomics Annual*, vol. 14, pp. 213–264. National Bureau of Economic Research.

Huggett, M. (1993): "The Risk Free Rate in Heterogeneous-Agent Incomplete-Insurance Economies," *Journal of Economic Dynamics and Control*, 17(5-6), 953–969.

Judd, K. L. (1985): "The Law of Large Numbers with a Continuum of IID Random Variables," *Journal of Economic Theory*, 35(1), 19–25.

Kaplan, G., and G. L. Violante (2014a): "A Model of the Consumption Response to Fiscal Stimulus Payments," *Econometrica*, 82(4), 1199–1239.

——— (2014b): "The Wealthy Hand-to-Mouth," *Brookings Papers on Economic Activity*, Spring, 77–138.

Kevin M. Murphy, F. W. (1992): "The Structure of Wages," *The Quarterly Journal of Economics*, 107(1), 285–326.

Kiyotaki, N., and J. Moore (2005): "2002 Lawrence R. Klein Lecture: Liquidity And Asset Prices," *International Economic Review*, 46(2), 317–349.

——— (2008): "Liquidity, Business Cycles and Monetary Policy," Working paper, Princeton University.

Kocherlakota, N. (2009): "Bursting Bubbles: Consequences and Cures," Working paper, University of Minnesota.

Krebs, T. (2006): "Recursive Equilibrium in Endogenous Growth Models with Incomplete Markets," *Economic Theory*, 29(3), 505–523.

Krusell, P., T. Mukoyama, and A. A. J. Smith (2011): "Asset Prices in a Huggett Economy," *Journal of Economic Theory*, 146(3), 812–844.

Krusell, P., and A. A. J. Smith (1998): "Income and Wealth Heterogeneity in the Macroeconomy," *Journal of Political Economy*, 106(5), 867–896.

Kubler, F., and K. Schmedders (2002): "Recursive Equilibria In Economies with Incomplete Markets," *Macroeconomic Dynamics*, 6, 284–306.

Kuhn, M. (2013): "Recursive Equilibria in an Aiyagari Style Economy with Permanent Income Shocks," *International Economic Review*, 54(3), 807–835.

Lagos, R., and R. Wright (2005): "A Unified Framework for Monetary Theory and Policy Analysis," *Journal of Political Economy*, 113(3), 463–484.

Le Grand, F., and X. Ragot (2016): "Incomplete Markets and Derivative Assets," *Economic Theory*, 62(3), 517–545.

Mankiw, N. G., and S. P. Zeldes (1991): "The Consumption of Stockholders and Nonstockholders," *Journal of Financial Economics*, 29(1), 97–112.

Mas-Colell, A., M. D. Whinston, and J. R. Green (1995): *Microeconomic Theory*. Oxford University Press.

Meyer, B., and J. Sullivan (2013): "Consumption and Income Inequaltiy and the Great Recession," *American Economic Review: Papers & Proceedings*, 103(3), 178–183.

Miao, J. (2006): "Competitive Equilibria of Economies with a Continuum of Consumers and Aggregate Shocks," *Journal of Economic Theory*, 128(1), 274–298.

Miao, J., and P. Wang (2017): "Bubbles and Credit Constraints," Working paper, Boston University.

Mukoyama, T., and A. Sahin (2006): "Costs of Business Cycles for Unskilled Workers," *Journal of Monetary Economics*, 53(8), 2179–2193.

Parker, J., and A. Vissing-Jorgensen (2009): "Who Bears Aggregate Fluctuations and How?," *American Economic Review*, 99(2), 399–405.

Polkovnichenko, V. (2004): "Limited Stock Market Participation and the Equity Premium," *Finance Research Letters*, 1(1), 24–34.

Scheinkman, J., and L. Weiss (1986): "Borrowing Constraints and Aggregate Economic Activity," *Econometrica*, 54(1), 23–45.

Shimer, R. (2005): "The Cyclical Behavior of Equilibrium Unemployment and Vacancies," *American Economic Review*, 95(1), 25–49.

Stokey, N. L., R. E. J. Lucas, and E. C. Prescott (1989): *Recursive Methods in Economic Dynamics*. Harvard University Press.

Uhlig, H. (1996): "A Law of Large Numbers for Large Economies," *Economic Theory*, 8(1), 41–50.

van Rooij, M., A. Lusardi, and R. Alessie (2011): "Financial Literacy and Stock Market Participation," *Journal of Financial Economics*, 101(2), 449–472.

Vissing-Jorgensen, A. (2002): "Towards an Explanation of Household Portfolio Choice Heterogeneity: Nonfinancial Income and Participation Cost Structures," Working Paper 8884, NBER.

Walentin, K. (2010): "Earnings Inequality and the Equity Premium," *B.E. Journal of Macroeconomics (Contributions)*, 10(1), 1–23.

Wen, Y. (2015): "Money, Liquidity and Welfare," *European Economic Review*, 76(Supplement C), 1–24.