Online Appendix for: Uncovering Asset Market Participation from Household Consumption and Income

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1 Proof of Proposition 1

For bond holdings, we compute the marginal rate of substitution $\mathbb{E}_t[\left(c^i_{x,t+1}/c^i_{x,t}\right)^{-\gamma_x}|x]$ at date t for a household of type x (conditional on its type). We obtain:

$$\begin{aligned}
&\mathbb{E}_t[\exp\{-\gamma_x \Delta \log c^i_{x,t+1}\}|x] = \\
&\quad = \frac{1}{\beta_x R^f_{t+1}} \mathbb{E}_t\left[\exp\left\{-\kappa_x \omega_{t+1} - \sigma_x \varepsilon^i_{t+1} - \psi_x \sqrt{\omega_{t+1}} u_{t+1}\right\}\right] \\
&\quad = \frac{1}{\beta_x R^f_{t+1}} \mathbb{E}_t\left[\exp\left\{\sigma_x^2/2 - \kappa_x \omega_{t+1} - \psi_x \sqrt{\omega_{t+1}} u_{t+1}\right\}\right] \\
&\quad = \frac{1}{\beta_x R^f_{t+1}} \mathbb{E}_t\left[\exp\left\{\sigma_x^2/2 + (\psi_x^2/2 - \kappa_x)\omega_{t+1}\right\}\right].
\end{aligned}$$

2 Proof of Proposition 2

For stock holdings, we compute the quantity $\mathbb{E}_t[R^s_{t+1}\left(c^i_{x,t+1}/c^i_{x,t}\right)^{-\gamma_x}|x]$

$$\begin{aligned}
&\mathbb{E}_t[\exp\{\log R^s_{t+1} - \gamma_x \Delta \log c^i_{x,t+1}\}|x] = \\
&\quad = \frac{1}{\beta_x} \mathbb{E}_t\left[\exp\left\{\mu\, \omega_{t+1} + \sqrt{\omega_{t+1}} u_{t+1} - \kappa_x \omega_{t+1} - \sigma_x \varepsilon^i_{t+1} - \psi_x \sqrt{\omega_{t+1}} u_{t+1}\right\}\right] \\
&\quad = \frac{1}{\beta_x} \mathbb{E}_t\left[\exp\left\{\sigma_x^2/2 + \mu\, \omega_{t+1} - \kappa_x \omega_{t+1} + (1-\psi_x)\sqrt{\omega_{t+1}} u_{t+1}\right\}\right] \\
&\quad = \frac{1}{\beta_x} \mathbb{E}_t\left[\exp\left\{\sigma_x^2/2 + (\mu - \kappa_x + 1/2 + \psi_x^2/2 - \psi_x)\omega_{t+1}\right\}\right] \\
&\quad = \frac{1}{\beta_x} \mathbb{E}_t\left[\exp\left\{\sigma_x^2/2 + (\mu + \frac{1}{2} - \psi_x + \frac{\psi_x^2}{2} - \kappa_x)\omega_{t+1}\right\}\right].
\end{aligned}$$

3 Monte Carlo results

We assess the finite sample accuracy of our estimation method by Monte Carlo simulation. The structural parameters are the II empirical parameter estimates reported in the main paper. These II estimates are then plugged into the limited participation model to generate 100 samples, of the same size as our empirical data. Finally, for each of the 100 data samples, we compute the II estimates of the 24 parameters.

Our results are summarized in the boxplots of Figures 1–2.[1] True parameter values are

[1] Boxes provide the first, second, and third quartiles of the II estimates and whiskers provide the farthest

reported with horizontal lines. These figures show that all parameters are correctly estimated since true values are covered by the intervals given by the whiskers of the boxplots. Most of the 24 parameters are very precisely estimated since the true values are inside the boxes. The only exceptions are κ_s, ψ_b, and ψ_s, which are less precisely estimated than the other parameters.

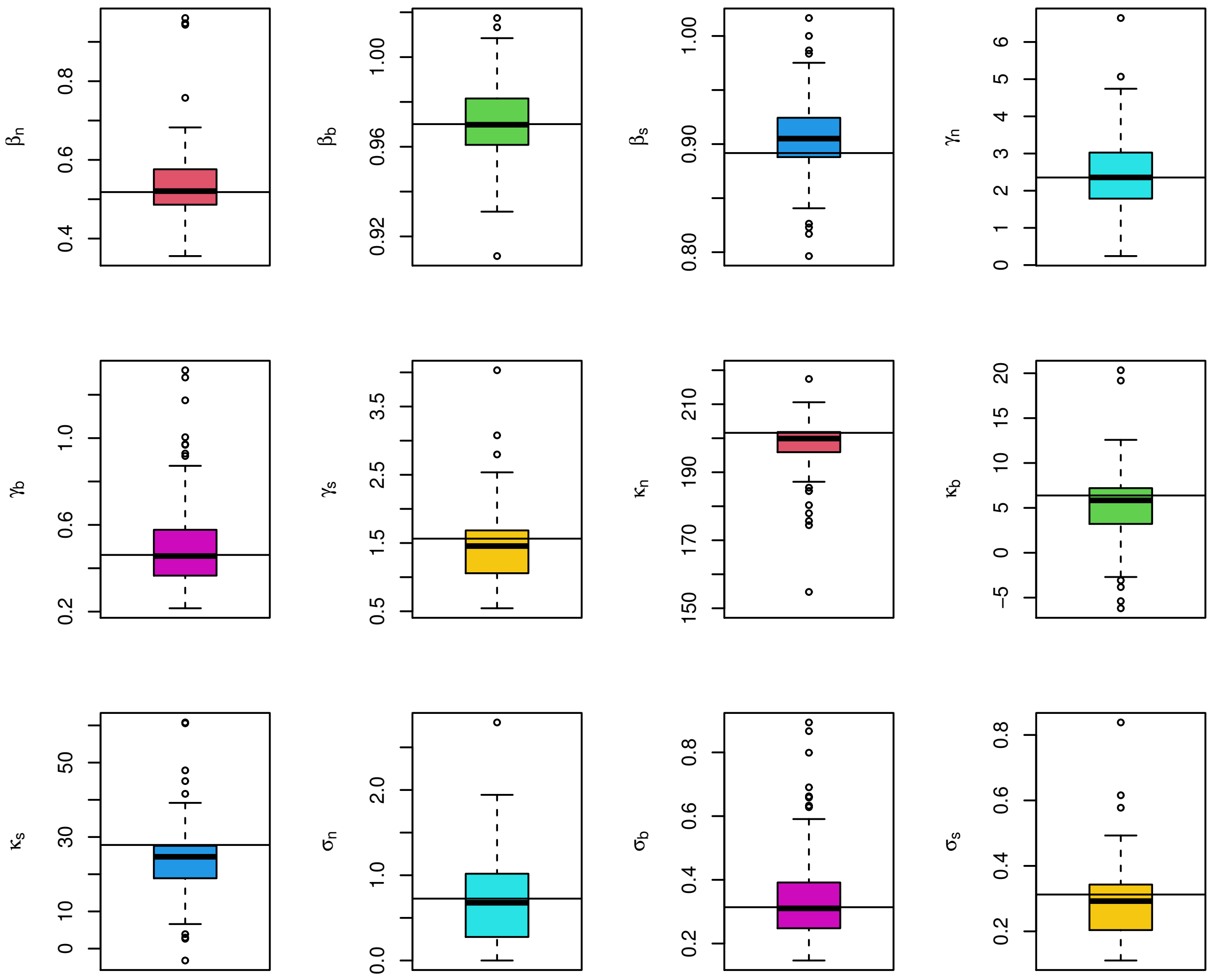

Figure 1: Boxplots of 100 MC estimates of the structural parameters β_n, β_b, β_s, γ_n, γ_b, γ_s, κ_n, κ_b, κ_s, σ_n, σ_b, and σ_s. True parameters are reported with horizontal lines.

II estimates that are within 1.5 times the interquartile range from the first and third quartiles. II estimates that are outside the whiskers are drawn by dots.

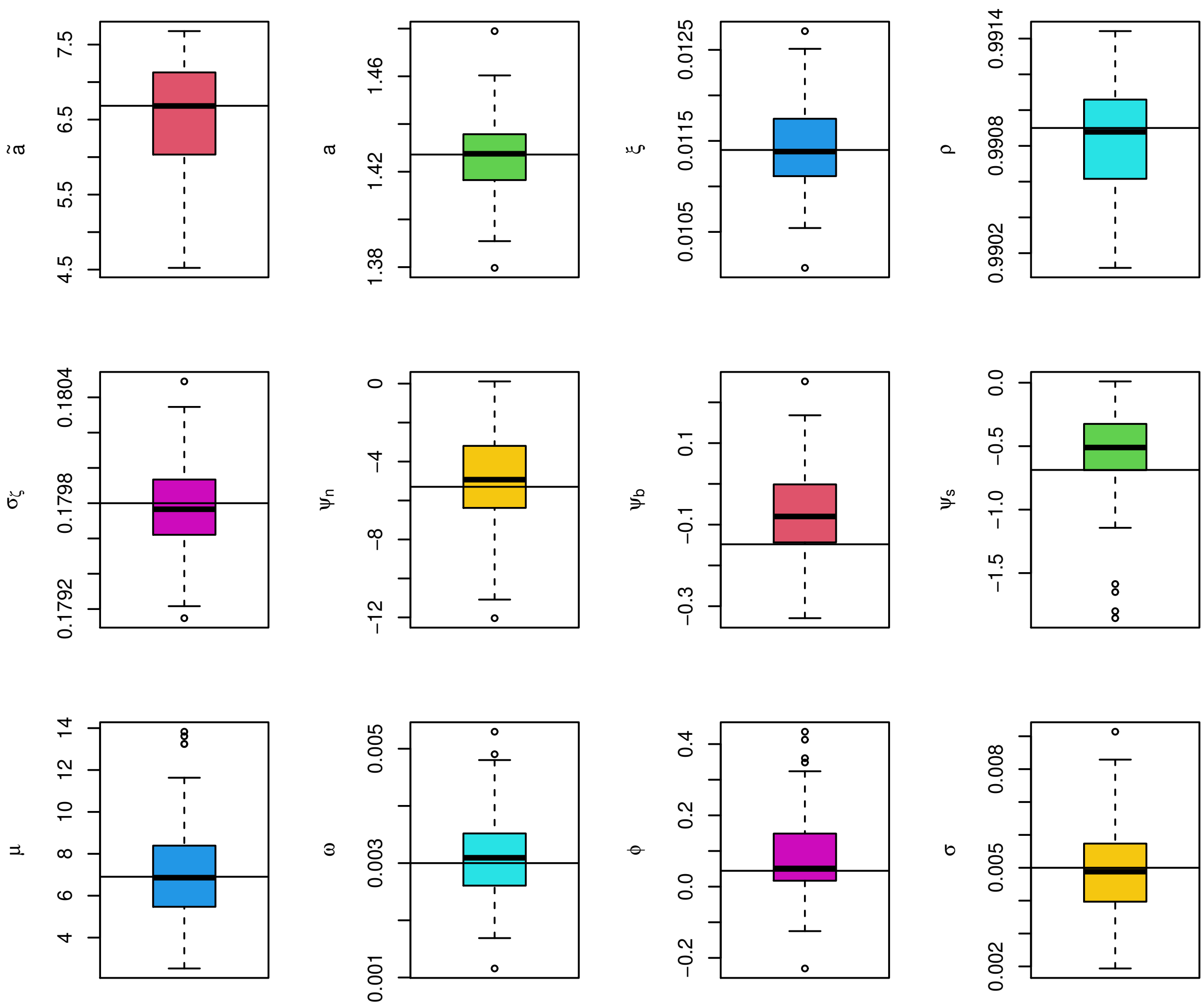

Figure 2: Boxplots of 100 MC estimates of the structural parameters $\tilde{a}$, a, ξ, ρ, σ_ζ, ψ_n, ψ_b, ψ_s, μ, ω, ϕ, and σ. True parameters are reported with horizontal lines.

4 Estimating the distribution of ω_t conditional on $l_{1:t} \equiv \{\log R^s_{t'} - \log R^f_{t'}\}_{t'=1,\ldots,t}$ using a particle filter

At each date t, the equity premium $l_t = \log R^s_t - \log R^f_t,\ t = 1, \ldots, T$ is assumed to be generated from the equity premium process with hidden states ω_t. The probability density function of the states ω_t given the observations $l_{1:t}$ is not available in closed form but can be easily obtained via a bootstrap particle filter (Gordon, Salmond and Smith 1993) as described below.

At date $t = 0$, generate particles $\{\tilde{\omega}_0^{(j)}\}_{j=1}^J$ using the stationary distribution $\mathcal{N}(\omega, \frac{\sigma^2}{1-\phi^2})$, where J is a fixed large positive integer. For $t \geq 1$, iterate the following three steps.

Step 1 (Sampling): For every $j = 1, \ldots, J$, simulate the filter forward as follows. Generate $\tilde{\omega}_t^{(j,*)}$ using $\tilde{\omega}_{t-1}^{(j)}$ by drawing $\tilde{\omega}_t^{(j,*)}$ from $\mathcal{N}[\omega + \phi(\tilde{\omega}_{t-1}^{(j)} - \omega), \sigma^2]$. For each $j = 1, \ldots, J$, calculate $\omega_t^{(j,*)} = \tilde{\omega}_t^{(j,*)}$ if $\tilde{\omega}_t^{(j,*)} \geq \omega$ and $\omega_t^{(j,*)} = \omega/(2\omega - \tilde{\omega}_t^{(j,*)})$ otherwise.

Step 2 (Correction): Given the new equity premium l_t, compute:

$$q_t^{(j)} = f(l_t|\omega_t^{(j,*)}) = \frac{1}{\sqrt{2\pi\omega_t^{(j,*)}}} \exp\left[-\frac{(l_t - \mu\,\omega_t^{(j,*)})^2}{2\omega_t^{(j,*)}}\right], \quad j = 1, \ldots, J\,,$$

Step 3 (Selection): For each $j = 1, \ldots, J$, draw $\tilde{\omega}_t^{(j)}$ from $\tilde{\omega}_t^{(1,*)}, \ldots, \tilde{\omega}_t^{(J,*)}$ with importance weights $r_t^{(1)}, \ldots, r_t^{(J)}$, where $r_t^{(j)} = q_t^{(j)} / \sum_{j'=1}^J q_t^{(j')}$. For each $j = 1, \ldots, J,$, calculate $\omega_t^{(j)} = \tilde{\omega}_t^{(j)}$ if $\tilde{\omega}_t^{(j)} \geq \omega$ and $\omega_t^{(j)} = \omega/(2\omega - \tilde{\omega}_t^{(j)})$ otherwise.

At each date t, the set $\{\omega_t^{(j)}\}_{j=1}^J$ finitely estimates the conditional distribution of ω_t given $l_{1:t}$.

5 Estimating the distribution of ω_{t+1} conditional on $l_{1:t} \equiv \{\log R^s_{t'} - \log R^f_{t'}\}_{t'=1,\ldots,t}$ using a particle filter

To estimate the distribution of ω_{t+1} using equity premium observations until date t, simulate a set of particles $\{\tilde{\omega}_t^{(j)}\}_{j=1}^J$ following the algorithm described in Section 4. Then, generate $\tilde{\omega}_{t+1}^{(j,*)}$ using $\tilde{\omega}_t^{(j)}$ by drawing $\tilde{\omega}_{t+1}^{(j,*)}$ from $\mathcal{N}[\omega + \phi(\tilde{\omega}_t^{(j)} - \omega), \sigma^2]$. For each $j = 1, \ldots, J$, calculate $\omega_{t+1}^{(j,*)} = \tilde{\omega}_{t+1}^{(j,*)}$ if $\tilde{\omega}_{t+1}^{(j,*)} \geq \omega$ and $\omega_{t+1}^{(j,*)} = \omega/(2\omega - \tilde{\omega}_{t+1}^{(j,*)})$ otherwise.

The set $\{\omega_{t+1}^{(j,*)}\}$ finitely estimates the conditional distribution of ω_{t+1} given $l_{1:t}$.

6 Estimating the distribution of $\omega_{1:T}$ conditional on $l_{1:T} \equiv \{\log R_{t'}^{s} - \log R_{t'}^{f}\}_{t'=1,\ldots,T}$ using a particle smoother

The probability density function of the states ω_t given the observations $l_{1:T}$ is not available in closed form but can easily be obtained via a particle smoother (Godsill et al. 2004) as described below.

Step 1 (Particle filtering): Use the particle filter defined in Section 4 to obtain an approximate particle representation of $f(\tilde{\omega}_t|l_{1:t})$ at each date $t = 1, \ldots, T$. Denote these particles by $\{\tilde{\omega}_t^{(j)}\}_{t=1,\ldots,T}^{j=1,\ldots,J}$.

For $k = 1, \ldots, K$, replicate Steps 2-4, where K is a fixed large positive integer.

Step 2 (Positioning of the backward simulation): Choose $\tilde{\omega}_T^{(k,*)} = \tilde{\omega}_T^{(j)}$ with probability $1/J$.

Step 3 (Backward simulation): For $t = T-1, \ldots, 1$ and each $j = 1, \ldots, J$

(i) compute the importance weights

$$q_{t|t+1}^{(j,k)} = f(\tilde{\omega}_{t+1}^{(k,*)}|\tilde{\omega}_t^{(j)}) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{[\tilde{\omega}_{t+1}^{(k,*)} - (\omega + \phi(\tilde{\omega}_t^{(j)} - \omega)]^2}{2\sigma^2} \right\}, \quad j = 1, \ldots, J\,;$$

(ii) choose $\tilde{\omega}_t^{(k,*)} = \tilde{\omega}_t^{(j)}$ with probability $r_{t|t+1}^{(j,k)}$, where $r_{t|t+1}^{(j,k)} = q_{t|t+1}^{(j,k)} / \sum_{j'=1}^{J} q_{t|t+1}^{(j',k)}$.

Step 4 (Path drawing): $\tilde{\omega}_{1:T}^{(k,*)} = (\tilde{\omega}_1^{(k,*)}, \ldots, \tilde{\omega}_T^{(k,*)})$ is an approximate realization from $f(\tilde{\omega}_{1:T}|l_{1:T})$.

For each $k = 1, \ldots, K$, and $t = 1, \ldots, T$, calculate $\omega_t^{(k,*)} = \tilde{\omega}_t^{(k,*)}$ if $\tilde{\omega}_t^{(k,*)} \geq \omega$ and $\omega_t^{(k,*)} = \omega/(2\omega - \tilde{\omega}_t^{(k,*)})$ otherwise. For each $k = 1, \ldots, K$, $\omega_{1:T}^{(k,*)} = (\omega_1^{(k,*)}, \ldots, \omega_T^{(k,*)})$ is an approximate realization from $f(\omega_{1:T}|l_{1:T})$.

7 Simulated maximum likelihood estimation of μ, ω, ϕ and σ using a particle filter

We can estimate μ, ω, ϕ, and σ by maximizing the simulated log-likelihood function associated with the equity premium process:

$$\sum_{t=1}^{T} \log \left(\frac{1}{J} \sum_{j=1}^{J} q_t^{(j)} \right) ,$$

where $q_t^{(j)}$ are defined in the second step of the particle filtering algorithm in Section 4. In order to obtain a smoother objective function, we choose a large number of particles $J = 10^6$.

8 Euler tests

In Figure 3 (continuous lines), we report the sample means $\{\overline{E}_{x,t}^{y}\}_{t=1}^{T}\}$ obtained using the J particles $\{\omega_t^j\}_{j=1}^{J}$ for $x = n, b, s$ and $y = B, S$. In addition, we also plot (dotted lines) the 5% critical values for $\{\overline{E}_{x,t}^{y}\}_{t=1}^{T}\}$ associated with the tests (T1)–(T6). Bond Euler conditions are in the left panels, while stock equations are in the right panels.

For bonds, the quantities $\{E_{s,t}^{B}\}_{t=1,\dots,T}$ (top left panel) and $\{E_{b,t}^{B}\}_{t=1,\dots,T}$ (middle left panel) lie within the Euler acceptance regions and (T1) and (T2) hold for all t. In the bottom left panel, we see that $E_{n,t}^{B}$ is less than one but not significantly for all t. Note that the test in the bottom left panel is unilateral and not bilateral as it was in the top panels. We can therefore conclude that (T3) holds – though not significantly – at all dates t.

For stocks, $E_{s,t}^{S}$ (top right panel) is not significantly different from 1 and (T4) holds for all t. In the middle right panel, we see that $E_{b,t}^{S}$ is significantly greater than one and that (T5) holds for all t, thereby confirming the presence of a stock market participation cost. In the bottom right panel, $E_{n,t}^{S}$ is less than one, but not significantly, and (T6) holds, but not significantly, for all t. One conclusion of the tests in Figure 3 is that our estimation is consistent with our initial interpretation of household types: stockholders (who also hold bonds) are of type s, bondholders (who do not hold stocks) are of type b, and nonparticipants are of type n.

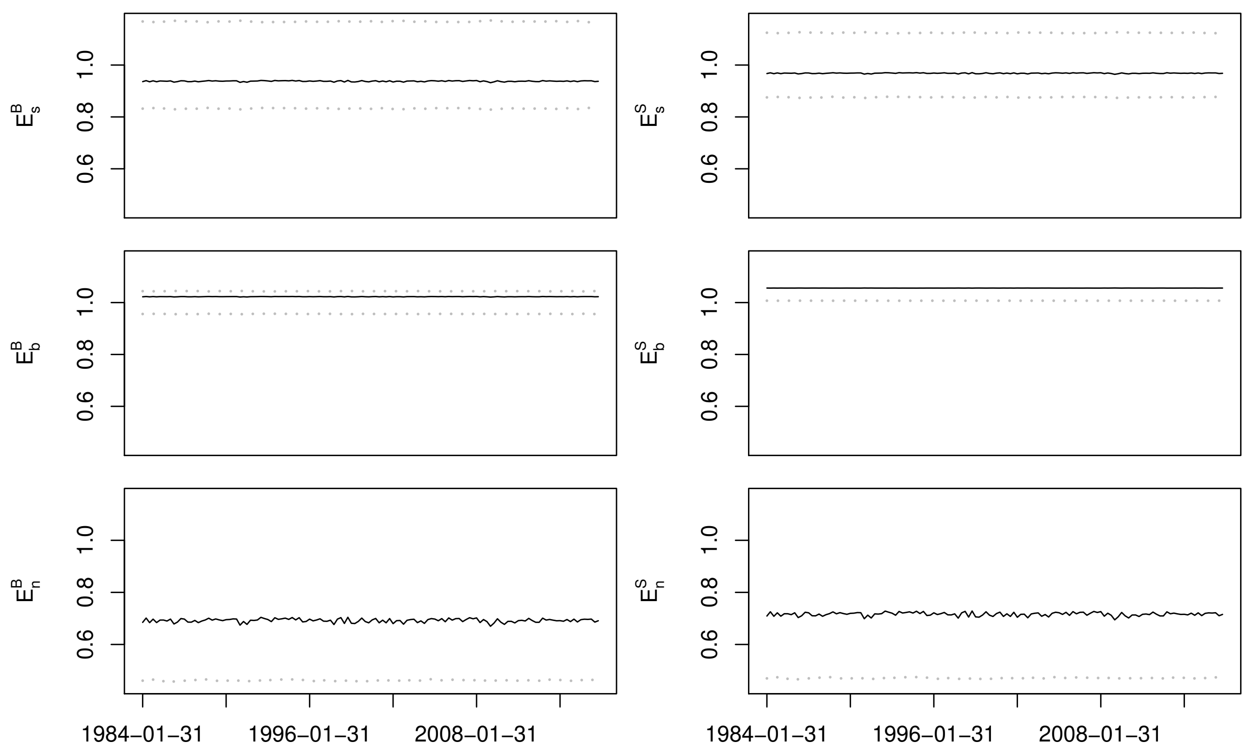

Figure 3: Test statistics (continuous lines) and associated critical values (dotted lines) for the six Euler conditions.

9 Unconditional Euler tests

We test whether the Euler conditions (T1)-(T6) in Section 4.3 of the main paper are satisfied for equity return data available from Kenneth French's Dartmouth website on 25 size/book-to-market sorted portfolios (Size/BM, Table 1), 10 long-run reversal portfolios (REV, Table 2), 25 size/investment portfolios (Size/INV, Table 3), 25 size/operating profitability portfolios (Size/OP, Table 4) and 10 industry portfolios (IND, Table 5). For each of these portfolios, we estimate the four equity premium parameters μ, ω, ϕ, and σ driving the equity premium dynamics. The estimation is performed using the simulated maximum likelihood described in Section 7 of the appendix. In Tables 2-5, we report the sample means and associated standard errors of $\{E^B_{x,t}\}_{t=1,\ldots,T}$ for bonds and $\{E^S_{x,t}\}_{t=1,\ldots,T}$ for stocks. The results are clear and strongly support the limited participation model at the 5% significance level. For bond holding, the three Euler conditions hold. For stock holding, the Euler condition for nonparticipants is significantly less than one most of the time. For stockholders, E^S_s is not significantly different from one most of the time. For bondholders, E^S_b is greater than one for most portfolios but not significantly. Overall, the tests for Euler conditions are consistent with the model and confirm that our estimated model properly isolates three categories of households: stockholders, bondholders, and nonparticipants.

Table 1: Unconditional Euler tests for Size/BM

	Euler Bond Conditions			Euler Stock Conditions		
Portfolio	$\overline{E}_n^B$	$\overline{E}_b^B$	$\overline{E}_s^B$	$\overline{E}_n^S$	$\overline{E}_b^S$	$\overline{E}_s^S$
SMALL LoBM	0.1314 (0.1706)	0.8963 (0.1102)	0.5810 (0.2520)	0.1373 (0.1757)	0.9038 (0.1053)	0.5902 (0.2497)
ME1 BM2	0.2388 (0.2493)	0.9490 (0.0752)	0.7049 (0.2113)	0.2486 (0.2550)	0.9725 (0.0587)	0.7241 (0.2031)
ME1 BM3	0.2572 (0.2181)	0.9745 (0.0447)	0.7692 (0.1428)	0.2697 (0.2231)	0.9976 (0.0315)	0.7904 (0.1342)
ME1 BM4	0.3159 (0.2606)	0.9812 (0.0458)	0.7928 (0.1487)	0.3310 (0.2656)	1.0152 (0.0238)	0.8217 (0.1350)
SMALL HiBM	0.2640 (0.2477)	0.9662 (0.0577)	0.7484 (0.1756)	0.2765 (0.2535)	0.9990 (0.0360)	0.7752 (0.1636)
ME2 BM1	0.1964 (0.2171)	0.9378 (0.0799)	0.6736 (0.2156)	0.2047 (0.2226)	0.9543 (0.0689)	0.6885 (0.2101)
ME2 BM2	0.3372 (0.2996)	0.9757 (0.0571)	0.7803 (0.1787)	0.3500 (0.3043)	1.0041 (0.0361)	0.8040 (0.1662)
ME2 BM3	0.3744 (0.2969)	0.9878 (0.0453)	0.8156 (0.1498)	0.3892 (0.3008)	1.0158 (0.0255)	0.8401 (0.1367)
ME2 BM4	0.3704 (0.2801)	0.9904 (0.0405)	0.8227 (0.1365)	0.3862 (0.2839)	1.0195 (0.0213)	0.8485 (0.1233)
ME2 BM5	0.2799 (0.2682)	0.9641 (0.0634)	0.7453 (0.1898)	0.2920 (0.2739)	0.9968 (0.0402)	0.7714 (0.1775)
ME3 BM1	0.2335 (0.2353)	0.9553 (0.0660)	0.7183 (0.1920)	0.2433 (0.2407)	0.9747 (0.0532)	0.7356 (0.1848)
ME3 BM2	0.3717 (0.3015)	0.9860 (0.0476)	0.8104 (0.1560)	0.3860 (0.3055)	1.0136 (0.0277)	0.8345 (0.1430)
ME3 BM3	0.3910 (0.2624)	0.9970 (0.0328)	0.8434 (0.1142)	0.4079 (0.2653)	1.0227 (0.0173)	0.8672 (0.1028)
ME3 BM4	0.3852 (0.2943)	0.9908 (0.0422)	0.8248 (0.1417)	0.4006 (0.2979)	1.0186 (0.0229)	0.8495 (0.1286)
ME3 BM5	0.3581 (0.3008)	0.9823 (0.0509)	0.7995 (0.1641)	0.3727 (0.3053)	1.0156 (0.0266)	0.8273 (0.1491)
ME4 BM1	0.3310 (0.3091)	0.9690 (0.0655)	0.7628 (0.1980)	0.3430 (0.3141)	0.9980 (0.0430)	0.7862 (0.1855)
ME4 BM2	0.4437 (0.3312)	0.9955 (0.0429)	0.8423 (0.1457)	0.4583 (0.3336)	1.0192 (0.0248)	0.8637 (0.1328)
ME4 BM3	0.4092 (0.3154)	0.9916 (0.0443)	0.8289 (0.1485)	0.4235 (0.3185)	1.0158 (0.0266)	0.8506 (0.1362)
ME4 BM4	0.4472 (0.3180)	0.9984 (0.0387)	0.8509 (0.1336)	0.4631 (0.3202)	1.0237 (0.0203)	0.8738 (0.1203)
ME4 BM5	0.2799 (0.2454)	0.9739 (0.0500)	0.7698 (0.1578)	0.2933 (0.2508)	1.0048 (0.0304)	0.7961 (0.1460)
BIG LoBM	0.4623 (0.3014)	1.0029 (0.0330)	0.8651 (0.1169)	0.4790 (0.3030)	1.0261 (0.0172)	0.8869 (0.1048)
ME5 BM2	0.5455 (0.3496)	1.0082 (0.0343)	0.8855 (0.1228)	0.5612 (0.3496)	1.0292 (0.0175)	0.9051 (0.1097)
ME5 BM3	0.5249 (0.3129)	1.0094 (0.0294)	0.8883 (0.1069)	0.5435 (0.3130)	1.0349 (0.0114)	0.9121 (0.0927)
ME5 BM4	0.4281 (0.3612)	0.9844 (0.0577)	0.8109 (0.1844)	0.4402 (0.3644)	1.0078 (0.0378)	0.8308 (0.1717)
BIG HiBM	0.3710 (0.3325)	0.9756 (0.0624)	0.7832 (0.1932)	0.3834 (0.3368)	1.0040 (0.0395)	0.8064 (0.1798)

Table 2: Unconditional Euler tests for REV

	Euler Bond Conditions			Euler Stock Conditions		
Portfolio	$\overline{E}_n^B$	$\overline{E}_b^B$	$\overline{E}_s^B$	$\overline{E}_n^S$	$\overline{E}_b^S$	$\overline{E}_s^S$
Loprior	0.2799	0.9456	0.7050	0.2899	0.9742	0.7260
	(0.2886)	(0.0889)	(0.2396)	(0.2944)	(0.0660)	(0.2296)
Prior 2	0.3688	0.9885	0.8175	0.3839	1.0162	0.8419
	(0.2833)	(0.0437)	(0.1448)	(0.2873)	(0.0246)	(0.1321)
Prior 3	0.4717	1.0016	0.8617	0.4878	1.0256	0.8838
	(0.3225)	(0.0366)	(0.1280)	(0.3242)	(0.0189)	(0.1148)
Prior 4	0.5377	1.0049	0.8753	0.5525	1.0272	0.8956
	(0.3678)	(0.0390)	(0.1371)	(0.3682)	(0.0203)	(0.1230)
Prior 5	0.5140	1.0052	0.8748	0.5299	1.0278	0.8958
	(0.3409)	(0.0357)	(0.1265)	(0.3416)	(0.0182)	(0.1131)
Prior 6	0.5301	1.0130	0.9000	0.5519	1.0409	0.9264
	(0.2699)	(0.0228)	(0.0847)	(0.2695)	(0.0059)	(0.0711)
Prior 7	0.5264	1.0143	0.9041	0.5485	1.0398	0.9287
	(0.2391)	(0.0192)	(0.0723)	(0.2386)	(0.0058)	(0.0611)
Prior 8	0.5272	1.0136	0.9018	0.5479	1.0373	0.9247
	(0.2548)	(0.0210)	(0.0787)	(0.2544)	(0.0077)	(0.0675)
Prior 9	0.4783	1.0058	0.8750	0.4955	1.0277	0.8958
	(0.2931)	(0.0299)	(0.1073)	(0.2942)	(0.0155)	(0.0959)
Hiprior	0.2932	0.9735	0.7702	0.3062	1.0029	0.7951
	(0.2607)	(0.0529)	(0.1659)	(0.2660)	(0.0334)	(0.1543)

Table 3: Unconditional Euler tests for Size/INV

Portfolio	Euler Bond Conditions $\overline{E}_n^B$	$\overline{E}_b^B$	$\overline{E}_s^B$	Euler Stock Conditions $\overline{E}_n^S$	$\overline{E}_b^S$	$\overline{E}_s^S$
SMALL LoINV	0.1908 (0.2187)	0.9287 (0.0896)	0.6540 (0.2321)	0.2007 (0.2257)	0.9723 (0.0592)	0.6835 (0.2212)
ME1 INV2	0.2807 (0.2318)	0.9783 (0.0436)	0.7819 (0.1412)	0.2950 (0.2369)	1.0090 (0.0256)	0.8086 (0.1298)
ME1 INV3	0.3082 (0.2183)	0.9882 (0.0335)	0.8118 (0.1136)	0.3251 (0.2227)	1.0203 (0.0166)	0.8407 (0.1020)
ME1 INV4	0.3371 (0.2763)	0.9831 (0.0465)	0.7998 (0.1515)	0.3509 (0.2806)	1.0089 (0.0292)	0.8226 (0.1402)
SMALL HiINV	0.1831 (0.2051)	0.9352 (0.0797)	0.6657 (0.2136)	0.1907 (0.2104)	0.9460 (0.0728)	0.6772 (0.2096)
ME2 INV1	0.2784 (0.2799)	0.9560 (0.0739)	0.7261 (0.2122)	0.2896 (0.2858)	0.9885 (0.0496)	0.7509 (0.2003)
ME2 INV2	0.3883 (0.2793)	0.9941 (0.0373)	0.8347 (0.1277)	0.4047 (0.2826)	1.0220 (0.0192)	0.8598 (0.1149)
ME2 INV3	0.3620 (0.2460)	0.9944 (0.0326)	0.8337 (0.1130)	0.3792 (0.2494)	1.0223 (0.0167)	0.8594 (0.1014)
ME2 INV4	0.3751 (0.3089)	0.9849 (0.0496)	0.8079 (0.1615)	0.3896 (0.3130)	1.0153 (0.0271)	0.8337 (0.1472)
ME2 INV5	0.2216 (0.2425)	0.9389 (0.0843)	0.6804 (0.2267)	0.2300 (0.2479)	0.9544 (0.0733)	0.6945 (0.2211)
ME3 INV1	0.2973 (0.2589)	0.9757 (0.0506)	0.7763 (0.1602)	0.3107 (0.2640)	1.0048 (0.0315)	0.8012 (0.1486)
ME3 INV2	0.4354 (0.2982)	0.9994 (0.0354)	0.8530 (0.1236)	0.4520 (0.3005)	1.0249 (0.0180)	0.8764 (0.1108)
ME3 INV3	0.4247 (0.2490)	1.0033 (0.0267)	0.8644 (0.0958)	0.4444 (0.2510)	1.0321 (0.0106)	0.8912 (0.0835)
ME3 INV4	0.3586 (0.2738)	0.9891 (0.0408)	0.8180 (0.1368)	0.3742 (0.2778)	1.0175 (0.0223)	0.8433 (0.1243)
ME3 INV5	0.2596 (0.2655)	0.9529 (0.0744)	0.7165 (0.2117)	0.2694 (0.2709)	0.9743 (0.0590)	0.7344 (0.2036)
ME4 INV1	0.3720 (0.3075)	0.9844 (0.0499)	0.8062 (0.1622)	0.3858 (0.3115)	1.0113 (0.0301)	0.8295 (0.1494)
ME4 INV2	0.4199 (0.2712)	1.0004 (0.0313)	0.8552 (0.1106)	0.4381 (0.2736)	1.0286 (0.0140)	0.8811 (0.0976)
ME4 INV3	0.4606 (0.3036)	1.0024 (0.0337)	0.8635 (0.1189)	0.4773 (0.3053)	1.0263 (0.0172)	0.8858 (0.1064)
ME4 INV4	0.4654 (0.3416)	0.9976 (0.0423)	0.8498 (0.1447)	0.4805 (0.3436)	1.0231 (0.0223)	0.8725 (0.1305)
ME4 INV5	0.2702 (0.2781)	0.9512 (0.0786)	0.7143 (0.2208)	0.2802 (0.2837)	0.9758 (0.0601)	0.7339 (0.2117)
BIG LoINV	0.4428 (0.2760)	1.0030 (0.0300)	0.8645 (0.1068)	0.4617 (0.2779)	1.0315 (0.0122)	0.8908 (0.0934)
ME5 INV2	0.5624 (0.2652)	1.0161 (0.0206)	0.9112 (0.0778)	0.5807 (0.2642)	1.0331 (0.0105)	0.9283 (0.0689)
ME5 INV3	0.5803 (0.3516)	1.0119 (0.0317)	0.8985 (0.1151)	0.5963 (0.3507)	1.0318 (0.0156)	0.9174 (0.1021)
ME5 INV4	0.5770 (0.3845)	1.0080 (0.0383)	0.8864 (0.1358)	0.5915 (0.3839)	1.0291 (0.0196)	0.9057 (0.1214)
BIG HiINV	0.3729 (0.3149)	0.9825 (0.0528)	0.8013 (0.1698)	0.3862 (0.3190)	1.0090 (0.0328)	0.8240 (0.1570)

Table 4: Unconditional Euler tests for Size/OP

	Euler Bond Conditions			Euler Stock Conditions		
Portfolio	$\overline{E}_n^B$	$\overline{E}_b^B$	$\overline{E}_s^B$	$\overline{E}_n^S$	$\overline{E}_b^S$	$\overline{E}_s^S$
SMALL LoOP	0.1704	0.9244	0.6414	0.1777	0.9385	0.6546
	(0.1997)	(0.0893)	(0.2286)	(0.2052)	(0.0801)	(0.2243)
ME1 OP2	0.2766	0.9839	0.7969	0.2920	1.0129	0.8233
	(0.2022)	(0.0343)	(0.1148)	(0.2068)	(0.0197)	(0.1049)
ME1 OP3	0.3300	0.9875	0.8113	0.3454	1.0153	0.8364
	(0.2492)	(0.0386)	(0.1294)	(0.2534)	(0.0219)	(0.1181)
ME1 OP4	0.2888	0.9815	0.7913	0.3034	1.0104	0.8170
	(0.2286)	(0.0404)	(0.1328)	(0.2334)	(0.0239)	(0.1220)
SMALL HiOP	0.2384	0.9554	0.7193	0.2489	0.9809	0.7404
	(0.2403)	(0.0669)	(0.1942)	(0.2460)	(0.0498)	(0.1853)
ME2 OP1	0.2163	0.9335	0.6687	0.2244	0.9492	0.6825
	(0.2416)	(0.0898)	(0.2356)	(0.2471)	(0.0786)	(0.2302)
ME2 OP2	0.3597	0.9822	0.7994	0.3732	1.0087	0.8222
	(0.3028)	(0.0513)	(0.1652)	(0.3070)	(0.0320)	(0.1529)
ME2 OP3	0.3627	0.9938	0.8320	0.3796	1.0216	0.8575
	(0.2509)	(0.0338)	(0.1164)	(0.2544)	(0.0175)	(0.1047)
ME2 OP4	0.3806	0.9858	0.8107	0.3951	1.0155	0.8361
	(0.3111)	(0.0491)	(0.1605)	(0.3151)	(0.0271)	(0.1463)
ME2 OP5	0.2671	0.9729	0.7659	0.2809	1.0068	0.7945
	(0.2343)	(0.0490)	(0.1544)	(0.2399)	(0.0282)	(0.1421)
ME3 OP1	0.2286	0.9351	0.6741	0.2370	0.9522	0.6886
	(0.2529)	(0.0906)	(0.2384)	(0.2584)	(0.0780)	(0.2325)
ME3 OP2	0.3984	0.9915	0.8279	0.4132	1.0169	0.8506
	(0.3048)	(0.0430)	(0.1444)	(0.3080)	(0.0249)	(0.1319)
ME3 OP3	0.3846	1.0007	0.8543	0.4029	1.0249	0.8775
	(0.2192)	(0.0249)	(0.0890)	(0.2216)	(0.0130)	(0.0797)
ME3 OP4	0.3582	0.9909	0.8234	0.3741	1.0183	0.8482
	(0.2634)	(0.0378)	(0.1280)	(0.2672)	(0.0207)	(0.1162)
ME3 OP5	0.3216	0.9804	0.7912	0.3362	1.0129	0.8188
	(0.2694)	(0.0478)	(0.1545)	(0.2744)	(0.0262)	(0.1410)
ME4 OP1	0.2582	0.9486	0.7070	0.2675	0.9678	0.7232
	(0.2693)	(0.0796)	(0.2217)	(0.2747)	(0.0655)	(0.2144)
ME4 OP2	0.3931	0.9876	0.8165	0.4072	1.0142	0.8397
	(0.3167)	(0.0482)	(0.1586)	(0.3203)	(0.0283)	(0.1455)
ME4 OP3	0.4514	1.0027	0.8640	0.4686	1.0267	0.8865
	(0.2898)	(0.0318)	(0.1128)	(0.2916)	(0.0161)	(0.1008)
ME4 OP4	0.4243	0.9965	0.8440	0.4404	1.0231	0.8679
	(0.3048)	(0.0386)	(0.1327)	(0.3075)	(0.0200)	(0.1194)
ME4 OP5	0.4102	0.9923	0.8310	0.4254	1.0198	0.8552
	(0.3131)	(0.0434)	(0.1459)	(0.3163)	(0.0233)	(0.1322)
BIG LoOP	0.3371	0.9579	0.7380	0.3462	0.9731	0.7515
	(0.3328)	(0.0813)	(0.2314)	(0.3372)	(0.0687)	(0.2241)
ME5 OP2	0.4626	0.9988	0.8529	0.4770	1.0191	0.8718
	(0.3319)	(0.0401)	(0.1382)	(0.3338)	(0.0247)	(0.1267)
ME5 OP3	0.4968	0.9978	0.8522	0.5102	1.0188	0.8711
	(0.3717)	(0.0460)	(0.1560)	(0.3732)	(0.0280)	(0.1430)
ME5 OP4	0.5989	1.0151	0.9096	0.6151	1.0325	0.9266
	(0.3334)	(0.0275)	(0.1018)	(0.3319)	(0.0142)	(0.0905)
BIG HiOP	0.4952	1.0086	0.8846	0.5130	1.0295	0.9047
	(0.2833)	(0.0269)	(0.0979)	(0.2839)	(0.0137)	(0.0872)

Table 5: Unconditional Euler tests for IND

	Euler Bond Conditions			Euler Stock Conditions		
Portfolio	$\overline{E}_n^B$	$\overline{E}_b^B$	$\overline{E}_s^B$	$\overline{E}_n^S$	$\overline{E}_b^S$	$\overline{E}_s^S$
NoDur	0.5062	1.0103	0.8904	0.5272	1.0388	0.9171
	(0.2759)	(0.0250)	(0.0919)	(0.2761)	(0.0074)	(0.0779)
Durbl	0.3161	0.9532	0.7250	0.3264	0.9811	0.7460
	(0.3195)	(0.0841)	(0.2352)	(0.3247)	(0.0611)	(0.2242)
Manuf	0.4658	0.9960	0.8452	0.4801	1.0199	0.8665
	(0.3508)	(0.0450)	(0.1522)	(0.3529)	(0.0256)	(0.1386)
Enrgy	0.3593	0.9913	0.8244	0.3748	1.0164	0.8475
	(0.2627)	(0.0374)	(0.1270)	(0.2663)	(0.0219)	(0.1159)
HiTec	0.2879	0.9389	0.6918	0.2971	0.9646	0.7105
	(0.3074)	(0.0974)	(0.2562)	(0.3129)	(0.0763)	(0.2473)
Telcm	0.3336	0.9938	0.8302	0.3500	1.0174	0.8527
	(0.2123)	(0.0285)	(0.0994)	(0.2156)	(0.0169)	(0.0906)
Shops	0.4687	0.9954	0.8436	0.4829	1.0201	0.8654
	(0.3568)	(0.0463)	(0.1561)	(0.3589)	(0.0259)	(0.1419)
Hlth	0.3716	1.0010	0.8546	0.3908	1.0264	0.8790
	(0.1943)	(0.0217)	(0.0783)	(0.1966)	(0.0108)	(0.0697)
Utils	0.5116	1.0160	0.9091	0.5341	1.0374	0.9306
	(0.1576)	(0.0117)	(0.0447)	(0.1572)	(0.0045)	(0.0384)
Other	0.3565	0.9845	0.8053	0.3703	1.0101	0.8279
	(0.2913)	(0.0474)	(0.1549)	(0.2954)	(0.0295)	(0.1431)

10 Deriving the participation cost

Consider at date t a household i of type b, that is, with $\tilde{h}_t^i = 1$ and $h_t^i = 0$. Its intertemporal utility $V_t^{b,i}$ is:

$$V_t^{b,i} = \frac{\left(c_t^{b,i}\right)^{1-\gamma_b}}{1-\gamma_b} + \beta_b \mathbb{E}_t \left[\frac{\left(c_{t+1}^{b,i}\right)^{1-\gamma_b}}{1-\gamma_b} \right] + \beta_b^2 \mathbb{E}_t \left[V_{t+2}^{b,i} \right],$$

$$= \frac{\left(c_t^{b,i}\right)^{1-\gamma_b}}{1-\gamma_b} \left(1 + \beta_b \mathbb{E}_t e^{(1-\gamma_b)\Delta \log c_{t+1}^{b,i}} \right) + \beta_b^2 \mathbb{E}_t \left[V_{t+2}^{b,i} \right]. \tag{10.1}$$

We consider the following thought experiment. Household i is constrained to participate in the stock market at date t. Household i remains endowed with individual preferences but must switch to the consumption growth process of stockholders, $\Delta \log c_{t+1}^{i,s}$. To compensate household i for this constrained stock market participation, it receives a flat amount $\tau_t^{b,i}$ at date t. We assume that this amount is fully consumed at date t. The intertemporal utility of this constrained household is denoted by $V_t^{b,i,s}$ and can be expressed as follows:

$$V_t^{b,i,s} = \frac{\left(c_t^{b,i} + \tau_t^{b,i}\right)^{1-\gamma_b}}{1-\gamma_b} \left(1 + \beta_b \mathbb{E}_t e^{(1-\gamma_b)\Delta \log c_{t+1}^{s,i}}\right) + \mathbb{E}_t \left[\beta^2 V_{t+2}^{b,i,s}\right]. \tag{10.2}$$

The compensation τ_t^i exactly offsets the forced participation if $V_t^{b,i,s} = V_t^{b,i}$. If we assume that the constrained participation has no effect after date $t+2$, the equality $V_t^{b,i,s} = V_t^{b,i}$ can be simplified using equations (10.1) and (10.2) as follows

$$\frac{\left(c_t^{b,i} + \tau_t^{b,i}\right)^{1-\gamma_b}}{1-\gamma_b} \left(1 + \beta_b \mathbb{E}_t e^{(1-\gamma_b)\Delta \log c_{t+1}^{s,i}}\right) = \frac{\left(c_t^{b,i}\right)^{1-\gamma_b}}{1-\gamma_b} \left(1 + \beta_b \mathbb{E}_t e^{(1-\gamma_b)\Delta \log c_{t+1}^{b,i}}\right),$$

or:

$$\frac{\tau_t^{b,i}}{c_t^{b,i}} = \frac{\left(1 + \beta_b \mathbb{E}_t e^{(1-\gamma_b)\Delta \log c_{t+1}^{b,i}}\right)^{\frac{1}{1-\gamma_b}}}{\left(1 + \beta_b \mathbb{E}_t e^{(1-\gamma_b)\Delta \log c_{t+1}^{s,i}}\right)^{\frac{1}{1-\gamma_b}}} - 1. \tag{10.3}$$

We will compute conditional expectation by conditioning on ω_{t+1} (more precisely, the filtration generated by (ω_t)). Formally:

$$\mathbb{E}_t \left[e^{(1-\gamma_b)\Delta \log c_{t+1}^{b,i}}\right] = \mathbb{E}_t \left[\mathbb{E}\left[e^{(1-\gamma_b)\Delta \log c_{t+1}^{b,i}} | \omega_{t+1}\right]\right].$$

Using the dynamics of the log-consumption growth, we have:

$$\begin{aligned}
(1-\gamma_b)\Delta \log c_{t+1}^{b,i} &= \frac{1-\gamma_b}{\gamma_b} \left\{\log \beta_b + \log(R_{t+1}^f) + \kappa_b \omega_{t+1} + \sigma_b \varepsilon_{t+1}^i + \psi_b \sqrt{\omega_{t+1}} u_{t+1}\right\}, \\
&\sim_{\omega_{t+1}} \mathcal{N}\left(\frac{1-\gamma_b}{\gamma_b}\left(\log \beta_b + \log(R_{t+1}^f) + \kappa_b \omega_{t+1}\right), \left(\frac{1-\gamma_b}{\gamma_b}\right)^2 \left(\sigma_b^2 + \psi_b^2 \omega_{t+1}\right)\right), \\
(1-\gamma_b)\Delta \log c_{t+1}^{s,i} &= \frac{1-\gamma_b}{\gamma_s} \left\{\log \beta_s + \log(R_{t+1}^f) + \kappa_s \omega_{t+1} + \sigma_s \varepsilon_{t+1}^i + \psi_s \sqrt{\omega_{t+1}} u_{t+1}\right\}, \\
&\sim_{\omega_{t+1}} \mathcal{N}\left(\frac{1-\gamma_b}{\gamma_s}\left(\log \beta_s + \log(R_{t+1}^f) + \kappa_s \omega_{t+1}\right), \left(\frac{1-\gamma_b}{\gamma_s}\right)^2 \left(\sigma_s^2 + \psi_s^2 \omega_{t+1}\right)\right).
\end{aligned}$$

In the above equations, $\sim_{\omega_{t+1}}$ denotes the law conditional on ω_{t+1}. We deduce:

$$\mathbb{E}_t\left[e^{(1-\gamma_b)\Delta \log c^{b,i}_{t+1}} | \omega_{t+1}\right] = \\ \exp\left(\frac{1-\gamma_b}{\gamma_b}\left(\log\beta_b + \log(R^f_{t+1}) + \kappa_b\omega_{t+1}\right) + \frac{1}{2}\left(\frac{1-\gamma_b}{\gamma_b}\right)^2\left(\sigma_b^2 + \psi_b^2\omega_{t+1}\right)\right),$$

and

$$\mathbb{E}_t\left[e^{(1-\gamma_b)\Delta \log c^{s,i}_{t+1}} | \omega_{t+1}\right] = \\ \exp\left(\frac{1-\gamma_b}{\gamma_s}\left(\log\beta_s + \log(R^f_{t+1}) + \kappa_s\omega_{t+1}\right) + \frac{1}{2}\left(\frac{1-\gamma_b}{\gamma_s}\right)^2\left(\sigma_s^2 + \psi_s^2\omega_{t+1}\right)\right).$$

We deduce:

$$\beta_b\mathbb{E}_t[e^{(1-\gamma_b)\Delta\log c^{b,i}_{t+1}}] = \beta_b^{\frac{1}{\gamma_b}}\left(R^f_{t+1}\right)^{\frac{1-\gamma_b}{\gamma_b}} e^{\left(\frac{1-\gamma_b}{\gamma_b}\right)^2\frac{\sigma_b^2}{2}}\,\mathbb{E}\left[e^{\left(\kappa_b + \frac{1-\gamma_b}{\gamma_b}\frac{\psi_b^2}{2}\right)\frac{1-\gamma_b}{\gamma_b}\omega_{t+1}}\right],$$
$$\beta_b\mathbb{E}_t[e^{(1-\gamma_b)\Delta\log c^{s,i}_{t+1}}] = \beta_b\beta_s^{\frac{1-\gamma_b}{\gamma_s}}\left(R^f_{t+1}\right)^{\frac{1-\gamma_b}{\gamma_s}} e^{\left(\frac{1-\gamma_b}{\gamma_s}\right)^2\frac{\sigma_s^2}{2}}\,\mathbb{E}\left[e^{\left(\kappa_s + \frac{1-\gamma_b}{\gamma_s}\frac{\psi_s^2}{2}\right)\frac{1-\gamma_b}{\gamma_s}\omega_{t+1}}\right].$$

Finally:

$$\frac{\tau_t^{b,i}}{c_t^{b,i}} = \frac{\left(1 + \beta_b^{\frac{1}{\gamma_b}}\left(R^f_{t+1}\right)^{\frac{1-\gamma_b}{\gamma_b}} e^{\left(\frac{1-\gamma_b}{\gamma_b}\right)^2\frac{\sigma_b^2}{2}}\mathbb{E}e^{\left(\kappa_b + \frac{1-\gamma_b}{\gamma_b}\frac{\psi_b^2}{2}\right)\frac{1-\gamma_b}{\gamma_b}\omega_{t+1}}\right)^{\frac{1}{1-\gamma_b}}}{\left(1 + \beta_b\beta_s^{\frac{1-\gamma_b}{\gamma_s}}\left(R^f_{t+1}\right)^{\frac{1-\gamma_b}{\gamma_s}} e^{\left(\frac{1-\gamma_b}{\gamma_s}\right)^2\frac{\sigma_s^2}{2}}\mathbb{E}e^{\left(\kappa_s + \frac{1-\gamma_b}{\gamma_s}\frac{\psi_s^2}{2}\right)\frac{1-\gamma_b}{\gamma_s}\omega_{t+1}}\right)^{\frac{1}{1-\gamma_b}}} - 1. \tag{10.4}$$

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